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FINAL EXAM

Allocate your time to the following four questions in proportion to the number of points available. If a question seems ambiguous, state why, sharpen it up and answer the revised question. Good luck!

1. (20) Consider a one-period economy, where many identical households have the following utility function:

$$u(c,l) = c - \frac{1}{2}l^2.$$

Each household's budget constraint is:

$$c \leq (1-\tau)\, wl,$$

where τ denotes the labor tax rate and w denotes the wage rate. The latter is technologically fixed at some positive number.

The government's budget constraint is

$$g \leq \tau wl,$$

where g denotes an exogenous spending requirement. The government chooses τ subject to its budge constraint, and it is benevolent in that its preferences coincide with those of the representative household.

In a Ramsey equilibrium, the government sets τ first and the private economy equilibrium occurs afterward. In a sustainable equilibrium, the government sets τ after equilibrium l is determined, but before equilibrium c is determined.

   (a) Define carefully a Ramsey equilibrium (e.g., spell out in detail the objects in an equilibrium, and define the problems of the agents and the consistency conditions.) Display the private sector allocation rules. Establish rigorously that there is only one Ramsey equilibrium.

(b) Define carefully a Sustainable equilibrium. Is there more than one Sustainable equilibrium? Provide a rigorous answer.

2. (30) Consider the following two-sector planning problem:

$$\max_{\{c_t, i_t, k_{1,t+1}, k_{2,t+1}, l_{1,t}, l_{2,t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{array}{c} c_t \leq z_t F(k_{1,t}, l_{1,t}) \\ i_t \leq q_t z_t F(k_{2t}, l_{2t}) \end{array}.$$

Here, k_{it} and l_{it} are capital and labor allocated to sector i, $i = 1, 2$. Assume that factors can be freely moved between sectors, subject to:

$$k_t \equiv k_{1,t} + k_{2,t}, \;\; l_{1,t} + l_{2,t} = l,$$

where k_t is the aggregate stock of capital given at the beginning of time t and l is the (fixed) amount of labor effort supplied by households. We also require

$$k_{t+1}, c_t \geq 0, k_0 > 0$$

and the identity

$$k_{t+1} \leq (1 - \delta) k_t + i_t.$$

The sequences $\{z_t, q_t\}_{t=0}^{\infty}$ are exogenously given. It is assumed that u is continuously differentiable, strictly increasing, and strictly concave, that F is continuously differentiable, strictly increasing in both arguments, homogeneous of degree one, and strictly quasiconcave, and that $\delta, \beta \in (0, 1)$.

(a) Show that a necessary condition for optimization is $\frac{k_{1t}}{l_{1t}} = \frac{k_{2t}}{l_{2t}}$, and that this implies the constraint set above can be replaced by

$$\begin{array}{c} c_t + \frac{i_t}{q_t} \leq z_t F(k_t, l) \\ k_{t+1} = (1 - \delta) k_t + i_t \text{ and } k_0 > 0 \\ c_t, k_{t+1} > 0. \end{array}$$

(b) Assume that the two productivity series obey:

$$z_t = \gamma_z^t \text{ and } q_t = \gamma_q^t,$$

where $\gamma_z \neq \gamma_q$ are each greater than unity. Suppose F has a Cobb-Douglas form, and $u(c) = c^{1-\sigma}/(1-\sigma)$, $0 < \sigma < 1$. Let a *steady-state growth path* be a situation in which c_t, k_t, and i_t are growing at a constant rate. Let γ_k and γ_c denote the steady-state growth rates of the capital stock and consumption, respectively. Develop equations that determine γ_k and γ_c as a function of the parameters of the problem.

(c) Scale the variables and show that the sequence planning problem can be expressed as a problem involving no growing variables.

(d) Express the non-growing, scaled, representation of the planning problem in functional equation form.

3. (30) Consider the following model. Preferences of the typical household are given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where

$$u(c_t) = \frac{c^{1-\nu} - 1}{1-\nu}, \ \nu > 0.$$

The household accumulates private capital using the following accumulation technology:

$$k_{p,t+1} = (1-\delta_p)k_{p,t} + i_{p,t},$$

where $k_{p,t}$ is the beginning-of-period t stock of capital, $i_{p,t}$ is period t gross investment, and $0 < \delta_p < 1$. The initial stock of capital, $k_{p,0}$, is given. The household is endowed with $n > 0$ units of labor time, and must satisfy $c_t, k_{p,t} \geq 0$.

The technology for producing output is

$$y_t = k_{gt}^{\gamma} k_{pt}^{\alpha} n_t^{(1-\alpha)}, \ 0 < \alpha < 1, \gamma \geq 0.$$

where k_{gt} is the per capita stock of government-provided capital. The household and firm take the sequence $\{k_{gt}\}$ as given and beyond their control.

The technology for accumulating government capital is:

$$k_{g,t+1} = (1 - \delta_g)k_{gt} + i_{gt},$$

where i_{gt} denotes per capita gross investment by the government, and $0 < \delta_g < 1$. The government finances investment by levying taxes, T_t, in the amount:

$$i_{gt} = T_t.$$

Taxes are levied on households in lump-sum form: each household must pay T_t, regardless of what decisions it takes regarding consumption, investment, or labor effort.

Finally, the resource constraint for this economy is:

$$c_t + i_{pt} + i_{gt} \leq y_t.$$

(a) Suppose the government chooses its sequence of public investment so that

$$k_{gt} = sk_{pt}, \ t \geq 1, \ s > 0.$$

Sharpen up the statement of the household and firm problems, and:

i. define a sequence of markets equilibrium.

ii. define a date 0 Arrow-Debreu equilibrium.

iii. define a recursive competitive equilibrium.

(b) Suppose $\gamma = 1 - \alpha$. Show that s can be chosen so that there is a steady-state balanced growth path for this economy, in which all quantity variables but employment display the same positive growth rate. Explain intuitively, why steady state growth is possible in this case, but is not when $\gamma = 0$.

(c) Display an optimization problem, the solution of which gives the efficient allocations for this economy. Write out the Euler equations for this problem.

(d) Suppose $\gamma = 1 - \alpha$. Suppose the economy is in a sequence-of-markets equilibrium, and that the government must optimally choose a sequence, i_{gt}, $t = 0, 1, 2, \ldots$. Would that sequence be consistent with the specification for public investment in (a)? Explain your answer.

4. (20) Let $g(k)$ denote the policy function in the simple growth model. Write down the model and a set of sufficient conditions on utility and preferences that imply the following property: for $k > 0$,

$$\begin{aligned} k &> k^* \Rightarrow k > g(k) > k^* \\ k &< k^* \Rightarrow k < g(k) < k^*, \end{aligned}$$

where k^* is the steady state of the model, *i.e.*, $k^* = g(k^*)$.