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411, Winter 2005

FINAL EXAM

Answer three of the following equally-weighted four questions. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have 1 hour and 55 minutes. Good luck!

3. Answer for question 3.

   (a) i. Sequence of markets equilibrium. At each date, t, the household maximizes discounted utility from then on:

$$\sum_{j=t}^{\infty} \beta^{j-t} u(c_j),$$

subject to a sequence of budget constraints:

$$c_j + i_{pj} \leq r_j k_{pj} + w_j n - T_j, \; j \geq t,$$

where w_j and r_j are market prices beyond the control of the household. The household uses its entire endowment of time for labor effort, n, because it does not value leisure. The firms choose n_t and $k_{p,t}$ such that profits are maximized, where profits are defined as follows:

$$k_{g,t}^{\gamma} n_t^{(1-\alpha)} k_{pt}^{\alpha} - w_t n_t - r_t k_t.$$

A sequence of markets equilibrium is a set of prices and quantities, $\{r_t, w_t; t \geq 0\}$, $\{y_t, c_t, n, i_{pt}, i_{gt}; t \geq 0\}$ and taxes, $\{T_t; t \geq 0\}$ such that

- given taxes and prices, the quantities solve the household problem.
- given the prices, the quantities solve the firm problem.
- given the quantities and a value of s, the government budget constraint is satisfied.

- the resource constraint is satisfied.

(b) the first order condition for the household is

$$u_{c,t} = \beta u_{c,t+1}[r_{t+1} + 1 - \delta_p],$$

and the firm sets $f_{k_p,t+1} = r_{t+1}$, where $f_{k_p,t+1}$ is the marginal product of private capital. Combining these, and taking functional forms into account:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\nu} = \beta[\alpha\left(\frac{nk_{g,t+1}}{k_{p,t+1}}\right)^{(1-\alpha)} + 1 - \delta_p].$$

Let g_c denote the gross growth rate of consumption in a balanced growth path. Then,

$$(g_c)^{\nu} = \beta[\alpha(ns)^{(1-\alpha)} + 1 - \delta_p].$$

Suppose g_c corresponds to some given positive net growth rate, ie., $g_c > 1$. Then,

$$s = \frac{1}{n}\left\{\frac{1}{\alpha}\left[\frac{g_c^{\nu}}{\beta} + \delta_p - 1\right]\right\}^{\frac{1}{1-\alpha}}.$$

The number in square brackets is positive, so that s is well defined. Thus the Euler equation is consistent with constant consumption growth in steady state. To fully answer the question, we need to establish (i) that the other equations - the household budget equation and the resource constraint - are also satisfied with a constant consumption growth rate and (ii) that the other quantity variables display positive growth too. Let g_g and g_p denote the gross growth rates of government and private capital, respectively. Then, the government's policy for choosing $k_{g,t}$ implies:

$$g_g = g_p = g,$$

say. Note that output can be written

$$k_{gt}^{(1-\alpha)}k_{pt}^{\alpha}n^{(1-\alpha)} = k_{gt}(k_{pt}/k_{gt})^{\alpha}n^{(1-\alpha)} = k_{gt}s^{\alpha}n^{(1-\alpha)}.$$

Divide the resource constraint by k_{gt}:

$$\frac{c_t}{k_{gt}} + g_{t+1} - (1-\delta_g) + g_{t+1} - (1-\delta_p) = s^{\alpha} n^{(1-\alpha)}.$$

So, in a constant growth steady state (i.e., $g_{t+1} = g$, constant) the consumption to public capital ratio is a constant, equal to the following:

$$s^{\alpha} n^{(1-\alpha)} + (1-\delta_g) + (1-\delta_p) + 2g.$$

But, the consumption to public capital ratio being constant implies:

$$g_c = g.$$

The household budget constraint is trivially satisfied, since it is equivalent with the resource constraint given the first order conditions of firms, linear homogeneity of the production function with respect to firms' choice variables, and the government budget constraint.

(c) The planner's problem is: choose $c_t, k_{g,t+1}, k_{p,t+1}$, $t \geq 0$ to maximize discounted utility. After substituting out consumption using the resource constraint, the problem becomes:

$$\max_{\{k_{g,t+1},k_{p,t+1}\}} \sum_{t=0}^{\infty} \beta^t u[k_{gt}^{(1-\alpha)} n^{(1-\alpha)} k_{pt}^{\alpha} + (1-\delta_g)k_{g,t} + (1-\delta_p)k_{p,t}$$
$$-k_{p,t+1} - k_{g,t+1}],$$

subject to the object in square brackets (consumption) being nonnegative at all dates, and to $k_{g,t}, k_{p,t} \geq 0$. The planner's first order conditions are:

$$\begin{aligned} u_{c,t} &= \beta u_{c,t+1}[f_{k_p,t+1} + 1 - \delta_p] \\ u_{c,t} &= \beta u_{c,t+1}[f_{k_g,t+1} + 1 - \delta_g], \end{aligned}$$

for $t = 0, 1, 2, ...$ With the functional forms:

$$\begin{aligned} \left(\frac{c_{t+1}}{c_t}\right)^{\nu} &= \beta[\alpha k_{g,t+1}^{\gamma} \left(\frac{n}{k_{p,t+1}}\right)^{(1-\alpha)} + 1 - \delta_p] \\ \left(\frac{c_{t+1}}{c_t}\right)^{\nu} &= \beta[\gamma (k_{g,t+1})^{\gamma-1} n^{(1-\alpha)} (k_{p,t+1})^{\alpha} + 1 - \delta_g]. \end{aligned}$$

Substituting out consumption using the resource constraint, these two equations represent a vector difference equation in k, k', k'', where $k = [k_g \ k_p]'$. There are many solutions to this equation that are consistent with the given initial condition, $k_0 = [k_{g,0} \ k_{p,0}]$. One can construct the whole family of solutions by indexing them by k_1: different values of k_1 give rise, by iterating on the euler equation, to different sequences of capital. Not all are optimal. Only the one solution that also satisfies the transversality condition is optimal. Thus, satisfying the Euler equation is not sufficient for an optimum.

(d) Setting $\gamma = 1 - \alpha$ and equating the planner's two first order conditions, we get:

$$
\begin{aligned}
& \beta[\alpha \left(\frac{nk_{g,t+1}}{k_{p,t+1}}\right)^{(1-\alpha)} + 1 - \delta_p] \\
= \ & \beta[(1-\alpha)n^{(1-\alpha)} \left(\frac{k_{p,t+1}}{k_{g,t+1}}\right)^{\alpha} + 1 - \delta_g],
\end{aligned}
$$

which requires that $\frac{k_{p,t+1}}{k_{g,t+1}}$ be a particular constant for $t = 0, 1, ...$ Call this constant s^*. By setting $s = s^*$ the government cannot do better, since this achieves the planner's optimum.

4. suppose that the production function, f, is strictly concave and utility is strictly concave and positive. Then, using the first order and envelope conditions,

$$\left[f'(k) - \frac{1}{\beta}\right][k - g(k)] \leq 0.$$

Then,

$$
\begin{aligned}
k &> k^* \Rightarrow f'(k) - \frac{1}{\beta} < 0 \Rightarrow g(k) \leq k \\
k &< k^* \Rightarrow f'(k) - \frac{1}{\beta} > 0 \Rightarrow g(k) \geq k.
\end{aligned}
$$

Here, strict concavity of f has been used. The last weak inequalities are in fact strict because, by property (iv), $k > 0$, $k \neq k^*$ implies $g(k) \neq k$.

Finally,

$$\begin{aligned} k &< k^* \Rightarrow g(k) < k^* \\ k &> k^* \Rightarrow g(k) > k^* \end{aligned}$$

by property (ii). ■