Homework #5
Economics 411
Due Thursday, November 2
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1. The following three sector exogenous growth model was proposed in Kongsamut, Rebelo and Xie (see their paper, 'Beyond Balanced Growth', on the course web site), to explain several key features of long-run (i.e., 1869-1990) growth: (i) K/Y is roughly constant, where K denotes the aggregate stock of capital, and Y denotes aggregate output, (ii) K grows at a roughly constant rate, (iii) the rate of return on capital is relatively constant, and (iv) resources have been reallocated out of agriculture and into services, while manufacturing has a relatively stable (truth is, manufacturing is falling a little) share in the economy.

   Consider the following technology. Agricultural output, A_t, is produced using the following production function:

$$A_t = B_A K_{At}^{\alpha} (N_{At} z_t)^{1-\alpha},$$

   where $B_A > 0$, $0 < \alpha < 1$, z_t denotes the state of technology, and K_A and N_A denote capital and labor allocated to agriculture. The state of technology evolves according:

$$z_t = exp(g) z_{t-1}, \; g > 0.$$

   The manufacturing sector produces output that can be converted into consumption goods, C_t, or capital goods:

$$C_t + K_{t+1} - (1-\delta) K_t = K_{Mt}^{\alpha} (N_{Mt} z_t)^{1-\alpha},$$

   in obvious notation. Finally, services, S_t, are produced according to:

$$S_t = B_S K_{St}^{\alpha} (N_{St} z_t)^{1-\alpha}.$$

   Suppose that at a point in time, households supply K_t units of capital to the capital rental market and 1 unit of labor to the labor markets, so that clearing in these markets requires:

$$N_{At} + N_{Bt} + N_{Mt} = 1, \; K_{At} + K_{Bt} + K_{Mt} = K_t.$$

Prices are denominated in units of manufactured goods, so that the price of a manufactured good is unity. Let the price, in units of manufactured goods, of an agricultural good, be P_{At}. Let the price of a service be P_{St}. Finally, let r_t and w_t denote the rental rate and wage rate. Suppose that the three technologies are operated by competitive firms.

(a) Show that competitive behavior by firms implies: $K_{At}/N_{At} = K_{Mt}/N_{Mt} = K_{St}/N_{St} = K_t$, $P_{At} = 1/B_A$, $P_{St} = 1/B_S$,

$$C_t + K_{t+1} - (1-\delta)K_t + \frac{A_t}{B_A} + \frac{S_t}{B_S} = K_t^{\alpha} z_t^{1-\alpha}. \tag{1}$$

One can treat the latter as a 'reduced form' expression for the economy's resource constraint.

(b) Derive an expression for the rate of return on capital. If K_t grows at the rate, g, will the rate of return on capital be constant?

(c) Suppose preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \frac{\left[\left(A_t - \bar{A}\right)^{\eta} C_t^{\gamma} (S_t + \bar{S})^{\theta}\right]^{1-\sigma}}{1-\sigma}, \; \eta + \gamma + \theta = 1, \; \eta, \gamma, \theta, \sigma > 0.$$

i. Write out a budget constraint for households and define a sequence of markets equilibrium for this economy.

ii. Derive the household's intertemporal Euler equation associated with capital.

iii. Show that household optimization, together with the results for prices you derived above, imply:

$$\frac{\gamma(A_t - \bar{A})}{\beta C_t} = B_A, \; \frac{\gamma(S_t + \bar{S})}{\theta C_t} = B_S. \tag{2}$$

iv. Substitute out for $A_t - \bar{A}$ and $S_t + \bar{S}$ in terms of C_t in the household's intertemporal Euler equation, to get an expression in terms of the growth rate of C_t and the rate of return on capital alone.

v. Show that in general, there is no reason to expect the economy to converge to a 'balanced growth path', i.e., one in which A_t, C, S_t, K_t, Y_t all grow at a constant rate. (Here, $Y_t = K_t^{\alpha} z_t{}^{1-\alpha}$.) Hint: note that if you scale A_t, M_t, K_t, S_t by z_t and work with the scaled versions of (2) and (1) and the household's intertemporal Euler equation, you cannot get rid of z_t.

vi. Consider the following restriction:

$$\bar{A}B_S = \bar{S}B_A. \tag{3}$$

Show that in this case, the economy boils down to the one sector growth model with disembodied technical change. Hint: note that in this case, you can replace A_t and S_t in (1) by $A_t - \bar{A}$ and $S_t - \bar{S}$, respectively without changing (1). Then, define $a_t = A_t - \bar{A}$ and $s_t = S_t - \bar{S}$ everywhere and impose (2), that a_t and s_t are each proportional to M_t.

vii. Explain why it is that under (3), the following are true:

$$\begin{aligned} \frac{K_t}{z_t} &\rightarrow k^*, \ \frac{A_t - \bar{A}}{z_t} \rightarrow a^*, \ \frac{S_t + \bar{S}}{z_t} \rightarrow s^* \\ \frac{K_{t+1}}{K_t}, \frac{Y_{t+1}}{Y_t} &\rightarrow exp(g), \end{aligned}$$

where k^*, a^*, s^* are finite, positive, constants.

viii. Provide a simple formula for k^*.

ix. What happens to the rate of return on capital along a growth path? What happens to the distribution of employment between sectors along a growth path?

x. What does the model imply for $P_{k'}$, the consumption price of capital, along the growth path.

2. Consider an economy with capital of different vintages. At time t, the amount of capital of vintage τ, $k_{t,\tau}$, $\tau = 1, 2, 3, \ldots$, is

$$k_{t,\tau} = \gamma^{t-\tau}(1-\delta)^{\tau-1} i_{t-\tau},$$

where $\gamma > 1$, $0 < \delta < 1$, $i_{t-\tau}$ is the amount of investment, in time $t - \tau$ consumption units, applied in period $t - \tau$. Capital which has vintage

τ in period t has vintage $\tau+1$ in period $t+1$. Investment expenditures at time t, i_t, must all be applied to the latest vintage (for a model in which investment in old vintages is feasible and desirable, see Chari and Hopenhayn, JPE, 1991) and results in $k_{t+1,1} = \gamma^t i_t$ units of new-vintage period $t+1$ installed capital goods. Consider a given amount of investment, i. Note that this investment applied in period $t+1$ produces more new-vintage installed capital (i.e., $\gamma^{t+1} i$) than the same level of investment applied in period t (i.e., $\gamma^t i$). This reflects the assumption, $\gamma > 1$ which is designed to capture the notion that there is exogenous technical progress that is embodied in new capital equipment. Note that the efficiency of a particular vintage stays constant over time, it's just that the efficiency of each succeeding vintage is greater than the efficiency of the previous one.

Capital of each vintage is operated with labor to produce a homogeneous output good, $y_{t,\tau}$ according to the following production function:

$$y_{t,\tau} = k_{t,\tau}^{\alpha} n_{t,\tau}^{1-\alpha}, \; 0 < \alpha < 1, \; \tau = 1, 2, 3, \ldots .$$

Suppose there is a competitive market in capital of different vintages and in labor. Each vintage of capital has the same rental rate, r_t, since capital is measured in common efficiency units. Similarly, the wage rate is w_t.

(a) Show that a firm's profit maximizing choice of $n_{t,\tau}$ gives rise to the following relationships:

$$y_t = k_t^{\alpha} n_t^{1-\alpha}, \; (1-\alpha)\left(\frac{k_t}{n_t}\right)^{\alpha} = w_t, \; \alpha\left(\frac{k_t}{n_t}\right)^{\alpha-1} = r_t,$$

where

$$y_t = \sum_{\tau=1}^{\infty} y_{t,\tau}, \; k_t = \sum_{\tau=1}^{\infty} k_{t,\tau}, \; n_t = \sum_{\tau=1}^{\infty} n_{t,\tau}.$$

(Hint: refer to your class notes on the indeterminacy of firm size under constant returns to scale.)

(b) Show that 'aggregate capital', k_t, evolves as in the Sollow II model:

$$k_{t+1} = (1-\delta)k_t + \gamma^t i_t.$$

3. Consider the endogenous growth model discussed in class. One sector produces a homogeneous output good, which is transformed one-for-one into consumption and investment using a Cobb-Douglas production function:

$$c_t + k_{t+1} - (1-\delta)\, k_t = k_t^{\alpha} n_t^{1-\alpha}.$$

Another sector produces human capital according to the following accumulation equation:

$$h_{t+1} = h_t + \lambda\,(h_t - n_t)\,,$$

where $\lambda > 0$, $c_t \geq 0$, $k_{t+1} \geq (1-\delta)\, k_t$, $0 \leq n_t \leq h_t$, and h_0, k_0 are given. Preferences are:

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma},$$

$\gamma > 0$. To ensure boundedness, we require $\beta\,(1+\lambda)^{1-\gamma} < 1$. In class, the problem was reformulated in recursive form. It was shown that there are policy rules of the form, $x_{t+1} = f\,(x_t)\,,\ y_t = g\,(x_t)\,,$ where $x_t = k_t/h_t$ and $y_t = h_{t+1}/h_t$.

- Set $\alpha = 1/3$, $\delta = 0.10$, $\beta = 0.97$, $\lambda = 0.04$, $\gamma = 1.1$. Compute steady state values of x, y. How do these values change with α and with λ? Provide intuition.
- Develop a formula for the date t price (in consumption units) of a unit of human capital, h_{t+1}. Develop a formula for the period $t+1$ payoff associated with an extra unit of h_{t+1} (hint: the payoff is the maximal increase in consumption that is possible in period $t+1$, while leaving the consumption opportunities unchanged in periods $t+2$ and later).
- Suppose x_t is below its steady state, either because some physical capital has been destroyed, or because the general level of human capital rose (say, because of immigration of high-human capital people). Is the price of human capital, h_{t+1}, low or high? What about the period $t+1$ payoff of h_{t+1}? Provide economic intuition. Recall that the one-period rate of return on an asset is the period $t+1$ payoff divided by the period t price. Can you say what the rate of return on human capital is when x_t is low? Explain.

- Write down a set of functional equations that define the equilibrium policy functions, $x_{t+1} = f(x_t)$, and $y_t = g(x_t)$. Explain how you would use the perturbation method to develop Taylor series approximations to f and g.