Homework #7
Economics 411, fall 2006
Due Thursday, November 16
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1. (Dynamic Inefficiency in OG Models). Consider the overlapping generations model in which the utility of the generation born at t is

$$u(c_t^t, c_{t+1}^t) = \log(c_t^t) + \beta \log(c_{t+1}^t).$$

The young supply one unit of labor inelastically in period zero, and earn the competitive wage rate, w_t. They use their income to purchase the outstanding stock of capital, and when old they finance their consumption from the earnings of the accumulated capital. Thus, their budget constraint is

$$c_t^t + k_{t+1} \leq w_t, \; c_{t+1}^t \leq r_{t+1} k_{t+1}.$$

Note that capital depreciates completely in one period. Firms are competitive in the output market and hire capital and labor in competitive factor markets where the prices are r_t and w_t, respectively. Their production technology is $y = k^\alpha n^{1-\alpha}$, $0 < \alpha < 1$.

   (a) Define a sequence of markets equilibrium. Provide expressions for w_t and r_t in terms of k_t.

   (b) Consider a steady state equilibrium in which the aggregate stock of capital of capital is constant, k, the consumption of each period's young is constant, c^y, and the consumption of each period's old is a constant, c^o. Time starts up in period 0, with the initial old generation owning the (steady state) capital stock, which they sell to the period 0 young. Show that the equilibrium rate of return on capital is

$$r_{k,t} = \frac{\alpha}{1-\alpha} \frac{1+\beta}{\beta}, \text{ for all } t.$$

   Interpret this expression. Why is the interest rate infinite if $\beta = 0$? Why is it zero if $\alpha = 0$?

   (c) Show that, for parameter values where $r_{k,t} < 1$, the competitive equilibrium is inefficient. That is, prove the following: it is possible to deviate from the equilibrium consumption allocations by reallocating consumption between each period's old and the same period's young in a way that is compatible with the resource constraint and which makes everyone (i.e., the first generation, the second generation, the third, etc.) better off. How might the result be affected if there were a last date in the economy?

   (d) Identify parameter values for which the growth rate of household consumption deviates from the (zero) growth rate of aggregate consumption that occurs in the steady state equilibrium.

2. Consider the OLG model presented in class, in which sustained growth in the aggregate capital stock cannot occur in equilibrium. Suppose that b is large so that there is a strong incentive to save. Display a set of equations that allow you to compute a steady state equilibrium (i.e., one in which all old consume the same amount, all young compute the same amount, the capital stock is constant and so are the rental rate and wage rate) with positive growth in the consumption of each household.

3. Following is a deterministic economy composed of one representative, competitive household, and a representative, competitive firm. Preferences of the household are given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t), \;\; u(c, n) = \log c + \sigma \log(1 - n),$$

where c_t denotes consumption and n_t denotes hours worked. The household budget equation is

$$c_t + I_t \leq w_t n_t + r_t k_t,$$

where w_t is the wage rate and r_t is the rental rate on capital, k_t. Here, I_t denotes investment, which the household applies to increasing the stock of capital, using the following technology:

$$k_{t+1} = (1 - \delta)k_t + I_t, \;\; 0 < \delta < 1.$$

The representative firm has access to the following technology:

$$y = Y^{\gamma} k^{\alpha} n^{1-\alpha}, \;\; \gamma = 1 - \alpha, \;\; \alpha = 1/3,$$

where Y is economy-wide average output, and y, k, n are firm output, capital, and employment, respectively. Note that the firm has constant returns to scale in the variables that it controls directly. Note too, the 'externality' in this production function. If all other firms are producing a lot (i.e., Y is big), this raises the productivity of an individual firm. Firms maximize profits given r_t and w_t. In equilibrium, $Y = y$.

   (a) Define a sequence-of-markets equilibrium for this economy.

   (b) Show that the Euler equations for labor and capital are, respectively:

$$w_t = -\frac{u_{n,t}}{u_{c,t}}, \;\; u_{c,t} = \beta u_{c,t+1} \left[r_{t+1} + 1 - \delta \right],$$

   where $u_{x,t}$ is the partial derivative of u with respect to $x = c_t, n_t$. Describe the transversality condition, and sketch how you would prove the sufficiency for household optimization of the transversality conditions and the Euler equations. (Hint: the proof can involve a lot of messy algebra. You need not do this. But, do indicate in a precise way what strategy you would take for establishing the proof.)

(c) Show that in an equilibrium, the first order conditions for firms and the resource constraint are (hint: impose $y = Y$):

$$r_t = \alpha n_t^2,\ w_t = \gamma k_t n_t,\ c_t + k_{t+1} - (1-\delta)k_t = k_t n_t^2.$$

(d) Show that by combining the household and firm Euler equations, one obtains:

$$\frac{n_t^2 + 1 - \delta - \frac{\gamma}{\sigma} n_t(1-n_t)}{n_t(1-n_t)} = \beta \frac{\alpha n_{t+1}^2 + 1 - \delta}{n_{t+1}(1-n_{t+1})},\ t = 0, 1, 2, \ldots . \quad (1)$$

Let (1) be represented as $v(n_t, n_{t+1}) = 0$. Note that this implicitly defines a map from n_t to n_{t+1}. Show that this map is composed of two functions, $n_{t+1} = f_i(n_t)$, $i = 1, 2$, where $f_1 > f_2$ for all n_t. Display analytic expressions for these functions. (Hint: remember the formula for the roots of a second order polynomial.)

(e) The following proposition is true. 'Suppose a sequence, n_0, $n_1, \ldots$ is found which satisfies (1) and also has the property, $a \leq n_t \leq b$ for $a > 0$ and $b < 1$ for all t. Then that sequence corresponds to an equilibrium.' Sketch a proof of this proposition.

(f) Explain why there are two stationary (i.e., equilibria with n_t constant) equilibria for this economy.

(g) Let n^* be the greater of the two stationary equilibria just identified. Show that there is an interval, D, about n^* such that if $n_0 \in D$ and n_t, $t = 1, 2, \ldots$, solves (1), then the given sequence, n_t, $t \geq 0$ represents an equilibrium. (Hint: note that $|\partial f_2(n)/\partial n| < 1$ for $n = n^*$.) Note that, since D is an interval, we have identified a continuum of equilibria. Explain why there are likely to be a great many more equilibria than just these two.

(h) Show that the efficient allocations involve a constant value of n_t, $\bar{n}$, say. It turns out that $n^* < \bar{n}$. Give an economic interpretation of this.