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411-1, Fall 2006

MIDTERM EXAM

There are four questions. The number of points available for each question is indicated. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have 1 hour and 55 minutes. Good luck!

1. (25) Suppose the efficient allocations solve the following sequence problem:

$$v(x_0) = \max_{\{x_{t+1}\in\Gamma(x_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}),$$

where $F : A \rightarrow R$, $A = \{(x,y) : x \in X,\ y \in \Gamma(x)\}$. Consider the following assumptions on (Γ, F, X, β) :

(A4.3) X convex, $X \subseteq R^l$; $\Gamma : X \rightarrow X$ is a non-empty, compact and continuous correspondence;

(A4.4) $F : A \rightarrow R$, is a bounded, continuous function, with $0 < \beta < 1$;

(A4.5) F is strictly increasing in its first argument for each value of its second argument;

(A4.6) Γ is monotone.

(a) provide an informal argument why the function, v, in the sequence problem also satisfies the following functional equation:

$$v(x) = \max_{y\in\Gamma(x)} F(x,y) + \beta v(y).$$

What assumptions are required to establish rigorously that the function, v, in the sequence problem and the functional equation are the same?

(b) Prove that assumptions (A4.3)-(A4.6) guarantee that v is strictly increasing. The part of your proof that uses (a) need not be rigorous. The rest of the proof should be rigorous.

2. (30) Consider the following functional equation:

$$T(v) = \max_{0 \leq \lambda \leq A+1-\delta} \frac{[A+1-\delta-\lambda]^{(1-\sigma)}}{1-\sigma} + \beta\lambda^{(1-\sigma)}v.$$

Suppose $\sigma > 1$, $A > \delta$, and $\beta(A+1-\delta)^{1-\sigma} < 1$.

(a) Show: $T(v) = \infty$ for $v > 0$, $T(0) = \frac{[A+1-\delta]^{(1-\sigma)}}{1-\sigma}$.

(b) Show: the derivative of T at $v = v_0 < 0$ is:

$$\frac{dT(v_0)}{dv} = \beta\lambda(v_0)^{(1-\sigma)},$$

where

$$\lambda(v_0) = argmax_{0 \leq \lambda \leq A+1-\delta} \frac{[A+1-\delta-\lambda]^{(1-\sigma)}}{1-\sigma} + \beta\lambda^{(1-\sigma)}v_0.$$

(c) Show that T does not satisfy the conditions of Blackwell's theorem.

(d) What happens to $\lambda(v)$ as $v \to -\infty$?

(e) What does the graph of $T(v)$ versus v for $v \leq 0$ look like? Does it cross a 45^0 line drawn in the negative orthant? Draw the $T(v)$ function by hand, emphasizing its qualitative features.

(f) Explain, using the graph you just developed, why $T^j v_0 = v^*$ as $j \to \infty$, for every $v_0 < 0$. Also, explain why v^* such that $v^* = T(v^*)$, is unique.

3. (30) Consider the two-sector economy, in which consumption and new capital are produced according to the following technologies,

$$c_t = k_{ct}^a n_{ct}^{1-\alpha}, \ k_{t+1} - (1-\delta)k_t = z_t k_{it}^{\alpha} n_{it}^{1-\alpha},$$

respectively. The price, in units of date t consumption goods, of new capital goods is p_t. Firms in the two sectors are competitive in output, capital and labor markets and take the rental rate on capital, r_t, and the wage rate, w_t, as given. The investment good firm takes p_t as given. All these prices are denominated in units of the date t consumption good. Profits of the consumption good firms, denominated in consumption units, are $k_{ct}^a n_{ct}^{1-\alpha} - r_t k_{ct} - w_t n_{ct}$, and profits of the investment good firms is $p_t z_t k_{it}^a n_{it}^{1-\alpha} - r_t k_{it} - w_t n_{it}$.

(a) What is the restriction across r_t and w_t that must be satisfied in equilibrium? Assume from here on that that restriction is satisfied.

(b) Show that the capital-labor ratios in the two sectors are the same, i.e., $k_{ct}/n_{ct} = k_{it}/n_{it}$.

(c) Show that

$$c_t + i_t = k_t^{\alpha} n_t^{1-\alpha}, \text{ where } k_{t+1} - (1-\delta)k_t = z_t i_t,$$

and

$$k_t = k_{ct} + k_{it}, \ n_t = n_{ct} + n_{it},$$

and

$$\alpha \left(\frac{k_t}{n_t} \right)^{\alpha - 1} = w_t, \ (1-\alpha) \left(\frac{k_t}{n_t} \right)^{\alpha} = r_t.$$

4. (15) Consider an economy with the following technology:

$$c_t + k_{t+1} \leq A k_t + k_t^{\alpha}, \ 0 < \alpha < 1, \ A > 0,$$

with $k_0 > 0$ given. Also, we require $k_t, c_t \geq 0$ for all t. Prove that when $A < 1$, there is a $\bar{k} < \infty$ such that feasibility implies $k_t \leq \bar{k}$ for all t. Explain how you can find $\bar{k}$ and what the role in the result is of the assumption, $A < 1$.