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Rough sketch of FINAL EXAM answers

Allocate your time to the following four questions in proportion to the number of points available. If a question seems ambiguous, state why, sharpen it up and answer the revised question. Good luck!

Ask stuff about the factor-price frontier.

1. A representative, competitive firm produces a homogeneous final good, Y, using the following production function:

$$Y = \left[\int_0^1 y_i^{\lambda} di\right]^{\frac{1}{\lambda}}, \ 0 < \lambda < 1,$$

where y_i is the quantity of the i^{th} intermediate good. The final good producer is competitive in the market for y_i, where it faces a price, P_i. Each intermediate good is produced by a single monopolist which sets price, P_i, to maximize profits. The first order necessary condition for profit maximization by the representative final good producer is the one associated with:

$$\max_{y_i} Y - \int_0^1 P_i y_i di,$$

and the first order condition is:

$$\left(\frac{Y}{y_i}\right)^{1-\lambda} = P_i. \tag{1}$$

The i^{th} intermediate good producer takes (1) as its demand curve, with Y being an exogenous variable.

The i^{th}, $i \in (0,1)$ intermediate good is produced by a monopolist using the production function:

$$y_i = A k_i^{\alpha} N_i^{1-\alpha}, \ 0 < \alpha < 1,$$

where A is a parameter of technology, and k_i, N_i denote capital and employment, respectively. Suppose that each monopolist is a price taker in the factor markets, where r and w are the rental rate on capital and wage rate on labor, respectively.

Monopolists, $i \in (0, \gamma)$ are subject to a 'financial friction': they must borrow the funds to pay rk_i and wN_i at the beginning of the period. So, at the end of the period these firms owe the bank principle plus interest equal to $Rrk_i + RwN_i$, where R is a gross rate of interest (i.e., a number like 1.08 for '8 percent interest'). Monopolists with $i \in (\gamma, 1)$ face no financial friction and simply pay rk_i and wN_i at the end of the period for its factor payments.

With the Cobb-Douglas production function, marginal cost for a firm that pays rental rate $\tilde{r}$ and wage rate $\tilde{w}$ is:

$$MC(\tilde{r}, \tilde{w}) = \xi \tilde{r}^{\alpha} \tilde{w}^{1-\alpha},$$

where ξ is a constant. The end of period profits for intermediate good produces are

$$\begin{aligned} P_i y_i - MC(r, w) y_i, \text{ for } i &\in (\gamma, 1) \\ P_i y_i - R \times MC(r, w) y_i, \text{ for } i &\in (0, \gamma) . \end{aligned}$$

To solve the monopolist maximization problem, substitute the demand curve out of the profit function:

$$(Y)^{1-\lambda} y_i^{\lambda} - C_i y_i,$$

where C_i denotes marginal cost:

$$C_i = \begin{cases} R \times MC(r, w) & i \in (0, \gamma) \\ MC(r, w) & i \in (\gamma, 1) \end{cases},$$

which is beyond the control by the monopolist. The first order condition is:

$$\lambda (Y)^{1-\lambda} y_i^{\lambda-1} = C_i,$$

or,

$$P_i = \frac{1}{\lambda} C_i,$$

for all $i \in (0,1)$. Solving,

$$\frac{\lambda (Y)^{1-\lambda}}{C_i} = y_i^{1-\lambda},$$

or,

$$y_i^{\lambda} = \left(\frac{\lambda}{C_i}\right)^{\frac{\lambda}{1-\lambda}} Y^{\lambda}.$$

Let $i' \in (0,\gamma)$, $i \in (\gamma, 1)$, so

$$\frac{y_{i'}}{y_i} = \left[\frac{\left(\frac{\lambda}{R\times MC(r,w)}\right)^{\frac{\lambda}{1-\lambda}} Y^{\lambda}}{\left(\frac{\lambda}{MC(r,w)}\right)^{\frac{\lambda}{1-\lambda}} Y^{\lambda}}\right]^{\frac{1}{\lambda}} = \left(\frac{1}{R}\right)^{\frac{1}{1-\lambda}},$$

or,

$$y_i = y_{i'} R^{\frac{1}{1-\lambda}},$$

so the unconstrained producers, i, produce more than the unconstrained ones. Substituting,

$$\begin{aligned}
Y &= \left[\int_0^1 y_i^{\lambda} di\right]^{\frac{1}{\lambda}} \\
&= \left[(1-\gamma)\, y_i^{\lambda} + \gamma y_{i'}^{\lambda}\right]^{\frac{1}{\lambda}} \\
&= \left[(1-\gamma)\, y_{i'}^{\lambda} R^{\frac{\lambda}{1-\lambda}} + \gamma y_{i'}^{\lambda}\right]^{\frac{1}{\lambda}} \\
&= \left[(1-\gamma)\, R^{\frac{\lambda}{1-\lambda}} + \gamma\right]^{\frac{1}{\lambda}} y_{i'} \\
&= \left[(1-\gamma)\, R^{\frac{\lambda}{1-\lambda}} + \gamma\right]^{\frac{1}{\lambda}} A k_{i'}^{\alpha} N_{i'}^{1-\alpha} \\
&= \left[(1-\gamma)\, R^{\frac{\lambda}{1-\lambda}} + \gamma\right]^{\frac{1}{\lambda}} A \left(\frac{N_{i'}}{k_{i'}}\right)^{1-\alpha} k_{i'}
\end{aligned}$$

To understand how producers use the factor inputs, consider the cost minimization problem for a producer which pays $\tilde{r}$ for capital and $\tilde{w}$ for labor:

$$\min \tilde{r} k_i + \tilde{w} N + \upsilon \left[y_i - A k_i^{\alpha} N_i^{1-\alpha}\right].$$

The first order conditions associated with this minimization problem are:

$$\begin{aligned} \tilde{r} &= \upsilon\alpha\left(\frac{N_i}{k_i}\right)^{1-\alpha} \\ \tilde{w} &= (1-\alpha)\,\upsilon\left(\frac{N_i}{k_i}\right)^{-\alpha} \end{aligned}$$

and, the ratio implies

$$\frac{r}{w} = \frac{\alpha}{(1-\alpha)}\frac{N_i}{k_i},$$

for all $i \in (0,1)$. So,

$$\frac{N_i}{k_i} = \frac{N}{k},\ \ i \in (0,1),$$

where N and k are the aggregate quantities of capital and labor. From this we see that producers use the same capital to labor ratio. As a result,

$$\begin{aligned} k_i &= k_{i'}R^{\frac{1}{1-\lambda}} \\ N_i &= N_{i'}R^{\frac{1}{1-\lambda}}, \end{aligned}$$

so

$$\begin{aligned} k &= \gamma k_{i'} + (1-\gamma)\,k_i \\ &= \left[\gamma + (1-\gamma)\,R^{\frac{1}{1-\lambda}}\right]k_{i'} \\ N &= \left[\gamma + (1-\gamma)\,R^{\frac{1}{1-\lambda}}\right]N_{i'} \end{aligned}$$

Conclude

$$\left[(1-\gamma)\,R^{\frac{\lambda}{1-\lambda}} + \gamma\right]^{\frac{1}{\lambda}} A\left(\frac{N}{k}\right)^{1-\alpha}\left[\gamma + (1-\gamma)\,R^{\frac{1}{1-\lambda}}\right]^{-1} k$$

$$Y = \frac{\left[(1-\gamma)\,R^{\frac{\lambda}{1-\lambda}} + \gamma\right]^{\frac{1}{\lambda}}}{\left[(1-\gamma)\,R^{\frac{1}{1-\lambda}} + \gamma\right]}A\left(\frac{N}{k}\right)^{1-\alpha}k,$$

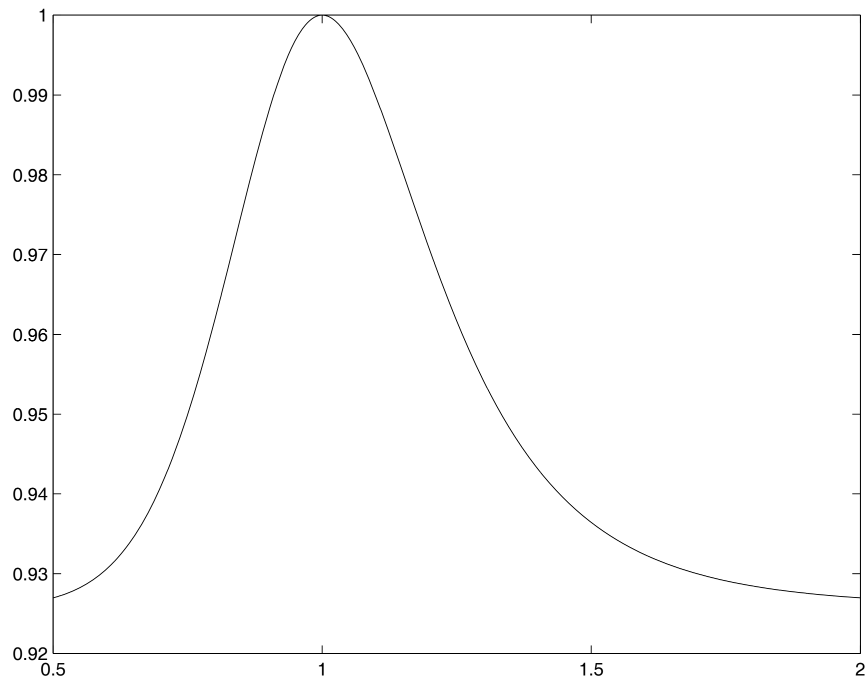
or,

$$Y = \frac{\left[(1-\gamma)\,R^{\frac{\lambda}{1-\lambda}} + \gamma\right]^{\frac{1}{\lambda}}}{\left[(1-\gamma)\,R^{\frac{1}{1-\lambda}} + \gamma\right]}Ak^{\alpha}N^{1-\alpha}.$$

From this, we see that estimated multifactor productivity is

$$\frac{Y}{k^{\alpha}N^{1-\alpha}} = \frac{\left[(1-\gamma)\,R^{\frac{\lambda}{1-\lambda}} + \gamma\right]^{\frac{1}{\lambda}}}{\left[(1-\gamma)\,R^{\frac{1}{1-\lambda}} + \gamma\right]} A.$$

This can move around in part because of movements in technology, A. However, it can also move around with movements in the endogenous variable, R. To see this, consider the case $\gamma = 0.5$, $\lambda = .9$ and R in the range 0.5-2:



Note how the function is maximized at $R = 1$ where resources are distributed equally (and, efficiently) among all producers. It is also the case that the inefficiency is quite small. For $R = 1.05$, TFP is about 0.9975, so that there is only a loss of about 0.25 percent in output.

2. Suppose there is a given total amount of some finite resource, X. This resource can be allocated among a range of inputs, $x(i)$, $i \in (0, M)$ subject to the resource constraint:

$$\int_0^M x(i)\,di \leq X.$$

Suppose the inputs can be converted into output according to the following technology:

$$y = n^{1-\alpha} \int_0^M x\left(i\right)^{\alpha} di, \; 0 < \alpha < 1.$$

Explain the sense in which the present environment is one in which there are gains from specialization.

answer: if the total resource, X, is allocated over a greater variety of inputs, $x\left(i\right), \; i \in \left(0, M\right)$, then more output is possible. Thus, suppose each input is operated at intensity

$$x\left(i\right) = \frac{X}{M},$$

so that the resource constraint is satisfied. Then, output is

$$y = n^{1-\alpha}\left(\frac{X}{M}\right)^{\alpha} M = n^{1-\alpha} X^{\alpha} M^{1-\alpha},$$

which is increasing in M, the range of inputs. As M increases, the range of inputs is greater, and this can be interpreted as greater specialization.

3. Consider the following competitive, two-period lived overlapping generations economy. People work and save when young. Their time endowment when young is unity. When, old households do not work and they pay for consumption out of the rent from capital accumulated while young. The population is constant, and the number of young born in each period is equal to the number of old who die in the same period. Let c_t^t and c_{t+1}^t denote the period t and $t+1$ consumptions, respectively, of agents born in period t. Let w_t denote the period t wage rate. Preferences of households are given by $u\left(c_t^t, c_{t+1}^t\right)$ and these are increasing and concave. The young supply one unit of labor inelastically. The budget constraint of the young and old are, respectively,

$$\begin{aligned} c_t^t + k_{t+1} &\leq w_t \\ c_{t+1}^t &\leq r_{t+1} k_{t+1} + \left(1-\delta\right) k_{t+1} \end{aligned}$$

where r_{t+1} denotes the rental rate of capital in period $t+1$ and w_t denotes the wage rate in period t. All quantities chosen by the household are required to be non-negative. The initial generation of old people owns the initial stock of capital, k_0, and simply consume the income from this stock. A representative, competitive firm chooses capital and labor in each period to maximize profits, using the following technology:

$$f(k_t, n) = b\left[\alpha k_t^{\rho} + (1-\alpha) n_t^{\rho}\right]^{\frac{1}{\rho}}, \ 0 < \rho < 1, \ b > 0,$$

where n_t denotes labor.

(a) Explain the presence of $(1-\delta) k_{t+1}$ on the right side of old people's budget constraint. Where does this income come from?

(b) Provide a formal, sequence of markets definition of equilibrium for this economy. Be sure to provide a resource constraint for the economy and explain how new capital is produced.

(c) Prove that growth is technologically feasible for some $b > 0$.

(d) Prove that growth (i.e., k_{t+1}/k_t for all $t > t^*$, some t^*) is not possible in equilibrium.

answer: (a) $(1-\delta) k_{t+1}$ is the amount of capital old people who saved k_{t+1} when young have left over at the end of the second period of their life. They sell this to capital producers who use $(1-\delta) k_{t+1}$ plus purchases in the goods market to construct k_{t+2} and sell this to the period $t+1$ young. (b) an equilibrium is a sequence of prices and quantities, $\{r_t, w_t, n_t, k_{t+1}, c_t^t, c_{t+1}^t, c_0^{-1}\}$, such that the quantities solve the household and firm problems, given the prices and goods and labor markets clear. Clearing in the goods market requires that the resource constraint be satisfied: $c_t^t + c_t^{t-1} + i_t \leq f(k_t, n)$, where i_t denotes the investment purchases of capital producers, who produce capital using the following technology:

$$k_{t+1} = (1-\delta) k_t + i_t.$$

(c) To see that growth is technologically feasible suppose $c_t^t = c_{t+1}^t = 0$ for all t and note from the resource constraint that as long a b is large enough, capital can grow forever:

$$k_{t+1} = f(k_t, 1) + (1-\delta) k_t.$$

This is so because for b large enough, $f_k(k_t, 1)$ will asymptote to as big a number as you want for b big enough. (d) Suppose that growth, in the sense described, does occur in equilibrium. Optimization by firms implies

$$w_t = b\left(\frac{f}{n}\right)^{1-\rho}(1-\alpha) = bf^{1-\rho}(1-\alpha),$$

using the fact, $n = 1$, in equilibrium. Then the share of income going to the young eventually goes to zero:

$$\frac{w_t}{f(k_t, n)} = \frac{bf^{1-\rho}(1-\alpha)}{f} = \frac{b(1-\alpha)}{f^{\rho}} = \frac{b^{1-\rho}(1-\alpha)}{[\alpha k_t^{\rho} + (1-\alpha)]} \to 0$$

as

$$c_t^t \leq w_t,$$

because $k_{t+1} \geq 0$.

4. answer:

   (a) i. Sequence of markets equilibrium. At each date, t, the household maximizes discounted utility from then on:

   $$\sum_{j=t}^{\infty} \beta^{j-t} u(c_j),$$

   subject to a sequence of budget constraints:

   $$c_j + i_{pj} \leq r_j k_{pj} + w_j n - T_j,\ \ j \geq t,$$

   where w_j and r_j are market prices beyond the control of the household. The household uses its entire endowment of time for labor effort, n, because it does not value leisure. The firms choose n_t and $k_{p,t}$ such that profits are maximized, where profits are defined as follows:

   $$k_{g,t}^{\gamma} n_t^{(1-\alpha)} k_{pt}^{\alpha} - w_t n_t - r_t k_t.$$

   A sequence of markets equilibrium is a set of prices and quantities, $\{r_t, w_t; t \geq 0\}$, $\{y_t, c_t, n, i_{pt}, i_{gt}; t \geq 0\}$ and taxes, $\{T_t; t \geq 0\}$ such that

- given taxes and prices, the quantities solve the household problem.
- given the prices, the quantities solve the firm problem.
- given the quantities and a value of s, the government budget constraint is satisfied.
- the resource constraint is satisfied.

(b) the first order condition for the household is

$$u_{c,t} = \beta u_{c,t+1}[r_{t+1} + 1 - \delta_p],$$

and the firm sets $f_{k_p,t+1} = r_{t+1}$, where $f_{k_p,t+1}$ is the marginal product of private capital. Combining these, and taking functional forms into account:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\nu} = \beta[\alpha\left(\frac{nk_{g,t+1}}{k_{p,t+1}}\right)^{(1-\alpha)} + 1 - \delta_p].$$

Let g_c denote the gross growth rate of consumption in a balanced growth path. Then,

$$(g_c)^{\nu} = \beta[\alpha(ns)^{(1-\alpha)} + 1 - \delta_p].$$

Suppose g_c corresponds to some given positive net growth rate, ie., $g_c > 1$. Then,

$$s = \frac{1}{n}\left\{\frac{1}{\alpha}\left[\frac{g_c^{\nu}}{\beta} + \delta_p - 1\right]\right\}^{\frac{1}{1-\alpha}}.$$

The number in square brackets is positive, so that s is well defined. Thus the Euler equation is consistent with constant consumption growth in steady state. To fully answer the question, we need to establish (i) that the other equations - the household budget equation and the resource constraint - are also satisfied with a constant consumption growth rate and (ii) that the other quantity variables display positive growth too. Let g_g and g_p denote the gross growth rates of government and private capital, respectively. Then, the government's policy for choosing $k_{g,t}$ implies:

$$g_g = g_p = g,$$

say. Note that output can be written

$$k_{gt}^{(1-\alpha)} k_{pt}^{\alpha} n^{(1-\alpha)} = k_{gt}(k_{pt}/k_{gt})^{\alpha} n^{(1-\alpha)} = k_{gt} s^{\alpha} n^{(1-\alpha)}.$$

Divide the resource constraint by k_{gt}:

$$\frac{c_t}{k_{gt}} + g_{t+1} - (1-\delta_g) + g_{t+1} - (1-\delta_p) = s^{\alpha} n^{(1-\alpha)}.$$

So, in a constant growth steady state (i.e., $g_{t+1} = g$, constant) the consumption to public capital ratio is a constant, equal to the following:

$$s^{\alpha} n^{(1-\alpha)} + (1-\delta_g) + (1-\delta_p) + 2g.$$

But, the consumption to public capital ratio being constant implies:

$$g_c = g.$$

The household budget constraint is trivially satisfied, since it is equivalent with the resource constraint given the first order conditions of firms, linear homogeneity of the production function with respect to firms' choice variables, and the government budget constraint.

(c) The planner's problem is: choose $c_t, k_{g,t+1}, k_{p,t+1}$, $t \geq 0$ to maximize discounted utility. After substituting out consumption using the resource constraint, the problem becomes:

$$\max_{\{k_{g,t+1}, k_{p,t+1}\}} \sum_{t=0}^{\infty} \beta^t u[k_{gt}^{(1-\alpha)} n^{(1-\alpha)} k_{pt}^{\alpha} + (1-\delta_g)k_{g,t} + (1-\delta_p)k_{p,t}$$
$$-k_{p,t+1} - k_{g,t+1}],$$

subject to the object in square brackets (consumption) being nonnegative at all dates, and to $k_{g,t}, k_{p,t} \geq 0$. The planner's first order conditions are:

$$\begin{aligned} u_{c,t} &= \beta u_{c,t+1}[f_{k_p,t+1} + 1 - \delta_p] \\ u_{c,t} &= \beta u_{c,t+1}[f_{k_g,t+1} + 1 - \delta_g], \end{aligned}$$

for $t = 0, 1, 2, ...$ With the functional forms:

$$\begin{aligned}
\left(\frac{c_{t+1}}{c_t}\right)^{\nu} &= \beta[\alpha k_{g,t+1}^{\gamma}\left(\frac{n}{k_{p,t+1}}\right)^{(1-\alpha)} + 1 - \delta_p] \\
\left(\frac{c_{t+1}}{c_t}\right)^{\nu} &= \beta[\gamma(k_{g,t+1})^{\gamma-1} n^{(1-\alpha)} \left(k_{p,t+1}\right)^{\alpha} + 1 - \delta_g].
\end{aligned}$$

Substituting out consumption using the resource constraint, these two equations represent a vector difference equation in k, k', k'', where $k = [k_g\ k_p]'$. There are many solutions to this equation that are consistent with the given initial condition, $k_0 = [k_{g,0}\ k_{p,0}]$. One can construct the whole family of solutions by indexing them by k_1: different values of k_1 give rise, by iterating on the euler equation, to different sequences of capital. Not all are optimal. Only the one solution that also satisfies the transversality condition is optimal. Thus, satisfying the Euler equation is not sufficient for an optimum.

(d) Setting $\gamma = 1 - \alpha$ and equating the planner's two first order conditions, we get:

$$\begin{aligned}
&\beta[\alpha\left(\frac{nk_{g,t+1}}{k_{p,t+1}}\right)^{(1-\alpha)} + 1 - \delta_p] \\
=\ &\beta[(1-\alpha)n^{(1-\alpha)}\left(\frac{k_{p,t+1}}{k_{g,t+1}}\right)^{\alpha} + 1 - \delta_g],
\end{aligned}$$

which requires that $\frac{k_{p,t+1}}{k_{g,t+1}}$ be a particular constant for $t = 0, 1, ...$ Call this constant s^*. By setting $s = s^*$ the government cannot do better, since this achieves the planner's optimum.