

Homework #2  
 Economics 411, Fall 2007  
 Due Wednesday, October 10.  
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1. This question asks you to redo Theorem 4.15 in a model that incorporates hours worked as a choice variable. Consider the following utility function:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t), \quad (1)$$

where  $c_t \geq 0$  and  $0 \leq n_t \leq 1$  denote date  $t$  consumption and employment, respectively. The resource constraint is:

$$c_t + k_{t+1} \leq f(k_t, n_t), \quad (2)$$

with  $k_t \geq 0$ . Here,  $u$  is strictly concave, differentiable, strictly increasing in  $c$  and strictly decreasing in  $n$ . Also,  $f$  is strictly increasing in each argument, is linearly homogeneous of degree one, differentiable and concave. Suppose  $c_t^*, k_t^* > 0$ ,  $0 < n_t^* < 1$  satisfy (2) at each  $t$  and  $k_0^* = \bar{k}_0$ , the given initial stock of capital. Also, these numbers satisfy the ‘Euler equations’:

$$\begin{aligned} u_c(c_t^*, n_t^*) &= \beta u_c(c_{t+1}^*, n_{t+1}^*) f_k(k_{t+1}^*, n_{t+1}^*), \\ u_c(c_t^*, n_t^*) f_n(k_t^*, n_t^*) + u_n(c_t^*, n_t^*) &= 0, \end{aligned}$$

for  $t = 0, 1, 2, \dots$ , and the ‘transversality condition’:

$$\lim_{T \rightarrow \infty} u_c(c_T^*, n_T^*) f_k(k_T^*, n_T^*) k_T^* = 0.$$

Here,  $u_c$  and  $u_n$  denote the derivatives of  $u$  with respect to its first and second argument, and similarly for  $f$ . Show that the given sequences  $\{c_t^*, k_t^*, n_t^*; t \geq 0\}$  produce the highest value of (1) within the set of sequences which satisfy (2) and the inequality constraints on consumption, labor and capital. (Hint: imitate the proof to Theorem 4.15 in the text.)

2. This question asks you to redo Theorem 4.15 in a model that takes into account uncertainty. Suppose that at each date  $t$  a random variable,  $s_t$ , is realized. It can take on any one of  $N$  possible values:

$$s(1), s(2), \dots, s(N).$$

Call  $s_t$  the state of nature at date  $t$ . Let  $s^t$  denote the history of states of nature up to time  $t$ :

$$s^t = (s_0, s_1, \dots, s_t).$$

At date 0,  $s_0$  is known. Thus, as of date 0, there is one possible history,  $s^0$ ; at date 1 there are  $N$  possible histories,  $s^1$ ; at date 2,  $N^2$  possible histories,  $s^2$ ; ... at date  $t$ ,  $N^t$  possible histories  $s^t$ , etc.

Let the probability of history  $s^t$  be denoted by  $\mu(s^t)$ .<sup>1</sup> Then, by the definition of a probability,

$$\mu(s^t) \geq 0, \text{ for all } s^t, \text{ and } \sum_{s^t} \mu(s^t) = 1, \text{ for every } t = 0, 1, 2, \dots,$$

where  $\sum_{s^t}$  denotes ‘the sum over all  $N^t$  possible values of  $s^t$ ’.

Let the  $N \times N$  matrix  $\pi$  be defined by:

$$\pi_{ij} = \text{Probability}[s_{t+1} = s(j) | s_t = s(i)].$$

- (a) Suppose  $v(s^t) = v^i$  if  $s_t = s(i)$ , for  $i = 1, \dots, N$ . That is, the value taken on by  $v(s^t)$  is a function only of the current state of nature. Let the  $N \times N$  matrix  $\pi^2$  be defined by  $\pi^2 = \pi \times \pi$ . Similarly, define  $\pi^3 = \pi^2 \pi$ , ...,  $\pi^k = \pi^{k-1} \pi$ .

- i. Prove that each row of  $\pi^k$  is a probability distribution (i.e., all elements of  $\pi^k$  are non-negative and  $\pi^k$  satisfies  $\pi^k \tau = \tau$ , where  $\tau$  is the  $N \times 1$  vector  $\tau = (1, 1, \dots, 1)'$ ).
- ii. Suppose  $s_0 = s(k)$ . Show that:

$$\sum_{t=0}^{\infty} \sum_{s^t} \mu(s^t) \beta^t v(s^t) = \kappa [I - \beta \pi]^{-1} v, \quad (3)$$

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<sup>1</sup>Strictly speaking, the notation should be  $\mu_t(s^t)$ . I omit the  $t$  subscript on the probability to keep from proliferating notation.

where  $v$  is an  $N \times 1$  column vector,  $v = (v^1, \dots, v^N)'$ , and  $\kappa$  is a  $1 \times N$  row vector with all zeros, except a one in the  $k^{th}$  entry. Recall the definition of a double sum:

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q_{ij} \equiv [q_{00} + q_{01} + q_{02} + \dots] + [q_{10} + q_{11} + q_{12} + \dots] + \dots,$$

where  $q_{ij}$  is an arbitrary set of numbers. (Hint: start by writing the expression on the left of the equality in (3) explicitly for  $t=0,1,2,\dots$ , and stare.)

iii. Show that:

$$\sum_{s^{t+1}} q(s^{t+1}) = \sum_{s^t} \sum_{s^{t+1}|s^t} q(s^{t+1}), \quad (4)$$

where  $s^{t+1} | s^t$  signifies ‘all possible histories  $s^{t+1}$ , given history  $s^t$  has occurred’ and  $q(s^t)$  is an arbitrary function of  $s^t$ . It’s enough to establish this for  $t = 1$  and  $N = 2$ .

(b) Consider the utility function:

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu(s^t) u(c(s^t)), \quad (5)$$

and resource constraint:

$$c(s^t) + k(s^t) \leq f(k(s^{t-1}), s_t). \quad (6)$$

Note that  $s_t$  shifts the production function. Assume  $u$  and  $f$  satisfy the same conditions stated above (with the obvious modifications to reflect the absence of hours worked from the problem!).

Suppose  $c^*(s^t)$ ,  $k^*(s^t) > 0$  satisfy (6) for all  $s^t$ ,  $t = 0, 1, 2, \dots$ , with  $k^*(s^{-1}) = \bar{k}_0$ , the given initial stock of capital. Suppose also that the ‘Euler equations’ are satisfied:

$$u_c(c^*(s^t)) = \beta \sum_{s^{t+1}|s^t} \frac{\mu(s^{t+1})}{\mu(s^t)} u_c(c^*(s^{t+1})) f_k(k^*(s^t), s_{t+1}),$$

for all  $s^t$ ,  $t \geq 0$ , and the ‘transversality condition’:

$$\lim_{T \rightarrow \infty} \sum_{s^T} \beta^T \mu(s^T) u_c(c^*(s^T)) f_k(k^*(s^{T-1}), s_T) k^*(s^{T-1}) \rightarrow 0.$$

Prove that  $\{c^*(s^t), k^*(s^t); t \geq 0, \text{ all } s^t\}$  yields the highest value of (5) within the set of all sequences that satisfy (6) and the non-negativity constraints on consumption and the stock of capital. (Hint: imitate the proof strategy of Theorem 4.15 as closely as you can, and make use of (4) when you group terms in the capital stock).

The Euler and transversality conditions are sometimes stated using the expectation operator:

$$u_{c,t} = \beta E_t u_{c,t+1} f_{k,t+1}$$

and

$$\lim_{T \rightarrow \infty} E_0 \beta^T u_{c,T} f_{k,T} k_T = 0,$$

where  $E_t$  denotes the mathematical expectation operator, conditional on information dated  $t$  and earlier (to understand the conditional expectation operator in the euler equation, recall that  $\frac{\mu(s^{t+1})}{\mu(s^t)}$  signifies the conditional probability of  $s^{t+1}$ , given  $s^t$ .)

3. (Boldrin-Montrucchio 1986). Consider the policy rule,  $g : [0, 1] \rightarrow [0, 1]$  :

$$g(x) = 4x(1 - x).$$

Draw this function, along with the 45 degree line, by hand in the unit box. Find an economy,  $(F, \Gamma, \beta, X)$ , for which the above function is the policy rule, where the economy satisfies all of our assumptions (i.e., assumptions A4.3-A4.9 in Stokey and Lucas). Here,  $x$  is the aggregate stock of capital at the beginning of the period and  $x'$  is its value at the end of the period.

Some hints: Recall,  $\Gamma, \beta \in (0, 1)$  and  $F$  satisfy

$$g(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta v(y),$$

where

$$v(x) = F(x, g(x)) + \beta v(g(x)).$$

Recall that the definition of a function or a correspondence must include a specification of the domain and range. Recall too that a function, say  $f(x, y)$ , is strictly concave iff:

$$f_{xx}(x, y) < 0, \quad f_{yy}(x, y) < 0, \quad f_{xx}(x, y)f_{yy}(x, y) - f_{xy}(x, y)^2 > 0,$$

for all  $(x, y)$  in the domain of  $f$ . Also, it is easy to verify that

$$g(x) = \arg \max_{y \in \Gamma(x)} \Psi(x, y),$$

when  $\Psi(x, y)$  is defined as follows:

$$\Psi(x, y) = -\frac{1}{2}y^2 + yg(x) - \frac{1}{2}Lx^2 + ax,$$

and  $L, a$  are known constants. Finally, note that  $v(x) = \Psi(x, g(x))$ , and use this to back out  $F$ . You can think of your task as having to identify values of  $a$  and  $L$  that ensure assumptions A4.3-A4.9 are satisfied. How many such values are there?