

Homework #3
Economics 411
Due Wednesday, October 24
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1. Suppose a planner chooses to maximize, by choice of c_0, c_1, c_2, \dots , the following expression:

$$u(c_0) + \delta[\beta u(c_1) + \beta^2 u(c_2) + \dots], u(c_t) = \log(c_t) \quad (1)$$

subject to

$$c_t = k_t^\alpha - k_{t+1}, 0 < \alpha < 1, c_t, k_{t+1} \geq 0, k_0 \text{ given,}$$

where $0 < \delta < \beta < 1$. When $\delta = 1$, this is the problem studied in exercises 2.2 and 4.9 in SL.

- (a) Let $k_{t+1} = g_t(k_t)$ denote the policy rule that solves this problem, $t = 0, 1, \dots$. From the perspective of period 0, the part of the problem from $t = 1$ and on looks exactly like the problem with $\delta = 1$. As a result, you know that the optimized value of $u(c_1) + \beta^2 u(c_2) + \dots$ has the form, $v(k_1)$, and you know how to compute $v(k_1)$ because it has a simple log-linear form. Use this to show that the optimal choice of k_1 has the form:

$$k_1 = g k_0^\alpha,$$

where g is a scalar. Derive an explicit formula relating g to the parameters of the model, β, α, δ . How does the saving rate from period $t = 1$ and on compare with the date 0 saving rate?

- (b) Is there a unique k^* with the property $k_t \rightarrow k^*$ as $t \rightarrow \infty$ for all k_0 ? Display a formula relating k^* to the parameters of the model.
- (c) Suppose $\beta = 1/1.03$, $\alpha = 1/3$, $\delta = 0.85$. Suppose $k_0 = k^*$. Display the values of $k_0, k_1, k_2, k_3, k_4, k_5$ that solve the problem as of date zero.

- (d) Now suppose that when date 1 occurs, the planner decides to reoptimize with respect to k_2, k_3, \dots . The initial condition for this problem is k_1 , the decision implemented by the planner last period. From the perspective of $t = 1$, the planner's preferences over c_t , $t \geq 1$ are as follows:

$$u(c_1) + \delta[\beta u(c_2) + \beta^2 u(c_3) + \dots]$$

and the resource constraint is as before. (Note how different the problem for $t \geq 1$ looks from the point of view of period 1 than it does from the point of view of period 0.) What values will the planner choose for k_1, k_2, k_3, k_4, k_5 ? If the planner chooses to reoptimize in this way every period, to what value will k_t actually tend?

- (e) Why are the values for k_2, k_3, k_4, k_5 chosen by the planner in date 1 different from the values planned for these variables as of date 0? Because of this difference, the problem is said to be *time inconsistent*. If δ had been set to one, we would not have had this problem. Why not?

Basically, the attitude of the planner is 'I'm very impatient today (the discount rate from period 0 to period 1 is $\beta\delta$), but I'll be less impatient tomorrow (the discount rate from period 1 to period 2 is β), so I'll consume a lot today and save a lot tomorrow.' Such an attitude is not time consistent because when tomorrow rolls around the planner says the same thing. In the end, the planner just ends up with a low capital stock. This type of model has been used to explain the behavior of smokers, who resolve that 'tomorrow I'll quit smoking, but tonight I'll just have one or two more'. It also has been used to explain the low US saving rate. The notion is that many people say, 'today I'll spend, and tomorrow I'll save', day after day. (See the papers of David Laibson, of Harvard.) Does the solution that we have used in (d) make any sense? Would a rational person really make decisions in the time-inconsistent way described there?

2. According to Theorem 4.15, the Euler and transversality conditions (equations (2) and (3) on page 98 of S-L) are sufficient for an interior

sequence of x_t 's to constitute an optimum of the SP problem. Necessity of the Euler equation is fairly obvious. Here is a sketch of an argument that establishes necessity of the transversality condition. This argument requires, in addition to the usual assumptions, $x_t = 0 \in X$, and $0 \in \Gamma(x_t)$ for $x_t \in X$. I sketch this argument below. Convert this sketch into a rigorous proof. In your proof you may use, without proof, any results from the book that you wish. However, you must be clear always about which assumptions you are using.

Let x_t^* , $t = 0, 1, 2, \dots$, denote a sequence of x_t 's that solve the SP. Then,

$$v(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta v(x_{t+1}^*), \quad t = 0, 1, 2, \dots,$$

where, of course, $x_0^* = x_0$, the initial condition. Let x_t be any other sequence that satisfies $(x_t, x_{t+1}) \in A$ (recall, A is the domain of F) for all t (does this imply that x_t is in the domain of v ?). Then,

$$v(x_t) - v(x_t^*) \leq F_1(x_t^*, x_{t+1}^*)(x_t - x_t^*).$$

The proof follows in a straightforward (not *trivial*) way by replacing x_t with 0, for all t , by multiplying both sides of this expression by β^t , and driving $t \rightarrow \infty$.

3. Consider an economy with a large number of identical households, each having preferences, $\sum_{t=0}^{\infty} \beta^t u(c_t)$. Suppose the resource constraint is $c_t + i_t \leq f(k_t)$, where $k_{t+1} = i_t + (1 - \delta)k_t$, f is strictly increasing and concave, $f'(k) \rightarrow \infty$ as $k \rightarrow 0$, $f'(k) \rightarrow 0$ as $k \rightarrow \infty$, $0 < \delta < 1$. Assume investment, i_t , is irreversible, i.e., it must be that $i_t \geq 0$. In addition, suppose $c_t, k_t \geq 0$ and that $k_0 > 0$ is given. Consider the functional equation associated with this problem:

$$\begin{aligned} v(k) &= \max_{k' \in \Gamma(k)} u(f(k) + (1 - \delta)k - k') + \beta v(k') \\ \Gamma(k) &= \{k' : (1 - \delta)k \leq k' \leq f(k)\}. \end{aligned}$$

- (a) State a set of assumptions on β and u that guarantee there is a unique, differentiable, concave v that solves the above functional equation. For each property of v , explain which assumptions are used to get it.

- (b) Show that monotonicity of $\Gamma(k)$, Assumption 4.6 in S-L, fails so that one of the conditions of Theorem 4.7 which guarantee strictly increasing v , is not satisfied.
- (c) Show that the feasible set for this economy satisfies the following ‘quasi-monotonicity property’: if $\tilde{k} \geq k$, then $\Gamma(k) + (1 - \delta)(\tilde{k} - k) \subseteq \Gamma(\tilde{k})$. Here, the sum of a set, say X , and a number, say a , is a new set, $X + a$, where $X + a \equiv \{x + a : x \in X\}$.
- (d) Show: v is an increasing function in k . (Hint: (i) following the basic strategy of the proof of Theorem 4.7, it’s enough to establish that the assumptions of Theorem 4.7 with the monotonicity assumption on Γ replaced by quasi-monotonicity guarantee Tw is increasing if w is; (ii) make use of the fact that if $k' \in \Gamma(k)$, then $\tilde{k}' = k' + (1 - \delta)(\tilde{k} - k) \in \Gamma(\tilde{k})$, $\tilde{k}' > k'$, and $f(\tilde{k}) + (1 - \delta)\tilde{k} - \tilde{k}' > f(k) + (1 - \delta)k - k'$.) Can you provide intuition for the fact that v is increasing even though Γ fails to satisfy monotonicity?
4. Do exercise 6.7a-f, pages 157-158 in S-L. Ex. 6.7d asks you to verify the existence of a two period cycle. For extra credit, do exercise 6.7g, which explores the ‘stability’ of that cycle.