

Homework #4  
 Economics 411  
 Due Wednesday, November 31  
 Christiano

1. Consider the neoclassical growth model studied in class, with  $\beta = 1/1.03$ ,  $\alpha = 1/3$ ,  $\delta = 0.10$ ,  $\gamma = 1$ , where preferences are given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma},$$

$$\frac{c^{1-\gamma} - 1}{1 - \gamma} \equiv \log(c), \quad \text{for } \gamma = 1,$$

and the aggregate resource constraint is given by:

$$c_t + k_{t+1} - (1 - \delta)k_t \leq k_t^\alpha.$$

What is the steady state value of  $k$ ?

- (a) How long does it take to close 95 percent of a gap between an initial value of the capital stock,  $k_0$ , and the steady state value? What is the answer if  $\gamma$  is increased to 2, or instead  $\delta$  is increased to 0.20, or if instead  $\alpha$  is increased to .9? (Each of these exercises are one-parameter perturbations on the baseline parameters.)
- (b) Consider the Solow model. This assumes that people save and invest a fixed fraction,  $s$ , of gross output,  $k^\alpha$  :

$$k_{t+1} - (1 - \delta)k_t = sk_t^\alpha$$

What value of  $s$  is required in order for the steady states of the neoclassical and Solow models to coincide? How much time does it take for 95 percent of the gap between  $k_0$  and steady state capital to be closed in the Solow model? Explain the different speeds of adjustment across the Solow and neoclassical models.

- (c) Consider the modification of the model in which there is a choice over how intensely to utilize the stock of capital. That is, let the resource constraint be

$$c_t + i_t \leq (u_t k_t)^\alpha,$$

where the services of capital are given by  $u_t k_t$  and  $u_t$  denotes the utilization rate of capital. Also,  $i_t$  denotes investment. Without additional restrictions, the efficient thing would be to set  $u_t = \infty$ . This is in fact not efficient because setting  $u_t$  requires investment goods,  $i_t^u$  :

$$i_t^u = a(u_t) k_t.$$

Here, the function  $a$  is increasing,  $a' > 0$ , and convex,  $a'' > 0$ . It is convenient to restrict the functional form of  $a$  so that in steady state,  $u_t = 1$ . The following functional form is useful:

$$a(u) = 0.5b\sigma_a u^2 + b(1 - \sigma_a)u + b((\sigma_a/2) - 1), \quad (1)$$

where  $\sigma_a$  is a parameter that controls the curvature of  $a$  at steady state,  $u_t = 1$  (curvature is defined as  $a''(u)u/a'(u)$ ). Also,  $b$ , is a function of the other parameters of the model, and it must be chosen so that  $u = 1$  in a steady state. The capital accumulation equation is:

$$k_{t+1} = (1 - \delta)k_t + i_t^k,$$

where  $i_t^k$  denotes investment in physical capital. Total investment is

$$i_t = i_t^u + i_t^k.$$

- i. Show that  $a(1) = 0$ ,  $a'(1) = b$ ,  $a''(1)/a'(1) = \sigma_a$ .
- ii. The optimal setting of  $u_t$  involves equating the marginal cost and marginal benefit of varying  $u_t$ . Display these marginal costs and benefits.
- iii. Derive an expression for the rate of return on capital (i.e., the marginal increase in consumption goods you can get in  $t$  by acquiring an additional unit of capital,  $k_{t+1}$ , in  $t$ ). Use it to show how an increase in the utilization of capital affects the rate of return on capital.
- iv. How does the utilization of capital vary with the stock of capital?
- v. Redo the calculations in (a) above for  $\sigma_a = 100$  and  $\sigma_a = 0.001$ . What happens to the speed of adjustment when curvature in the utilization cost function is reduced? Provide the intuition for your result.

2. Monopoly power in a variant of the neoclassical growth model. In the neoclassical growth model, there is only one good, so monopoly power would entail assuming that there is only one firm. To avoid this silly implication, we introduce a Dixit-Stiglitz technology which allows for the presence of monopoly power, without the implication that there is only one firm. This question is basically a review of simple price theory.

In this discussion, we first discuss the firm sector. Because this problem is static, it is convenient from the point of notation, to temporarily drop the time subscript,  $t$ . In discussing the firm sector, we take the aggregate supply of capital,  $k$ , and labor,  $n$ , as given. These cannot be determined until the households have been brought into the picture.

At first, the firm sector may seem algebra-intensive. However, at the very end the whole setup collapses into three simple equations, equations that are very similar to three analogous equations that characterize the firm sector in our decentralization of the neoclassical growth model.

Final goods,  $y$ , are produced by a representative, competitive firm, using a continuum of intermediate goods,  $x(i)$ ,  $i \in (0, 1)$  :

$$y = \left[ \int_0^1 x(i)^\lambda di \right]^{\frac{1}{\lambda}}, \quad 0 < \lambda < 1. \quad (2)$$

The price of the  $i^{th}$  intermediate good is  $p(i)$ , which the final good producer takes as given. The profits of the final good producer are:

$$\pi = y - \int_0^1 p(i)x(i)di,$$

where the price of the final good has been normalized at unity. The problem of the final good producer is to choose  $x(i)$ ,  $i \in (0, 1)$  to maximize profits subject to the budget constraint. The first order necessary condition for profit maximization is:

$$p(i) = y^{1-\lambda}x(i)^{\lambda-1}, \quad i \in (0, 1).$$

There is a single monopolist who produces  $x(i)$ . The monopolist sets its price,  $p(i)$ , and output,  $x(i)$ , treating the first order condition of the final good producer as its demand curve,  $p(i) = P(x(i), y)$ , where

$$P(x(i), y) = y^{1-\lambda}x(i)^{\lambda-1}. \quad (3)$$

The monopolist knows the value of  $y$  and correctly understands that its choice of  $x(i)$  has no impact on  $y$ . The monopolist uses capital and labor to produce  $x(i)$  using the following technology:

$$x(i) = k(i)^\alpha n(i)^{1-\alpha} = f(k(i), n(i)), \quad 0 < \alpha < 1, \quad (4)$$

where  $k(i)$  and  $n(i)$  are the capital and labor hired by the monopolist in the capital rental and labor markets, respectively. The monopolist is small in the factor markets, and so it takes the rental rate on capital,  $r$ , and the wage rate,  $w$ , as given. The monopolist's problem is to choose  $x(i)$  to maximize profits, subject to (3), the technology, (4), and the given  $r, w$ . It is convenient to derive an expression for the monopolist's cost function,  $C(x(i), w, r)$  :

$$C(x(i), w, r) = \min_{k(i), n(i)} rk(i) + wn(i),$$

subject to (4). In Lagrangian form,

$$C(x(i), w, r) = \min_{k(i), n(i)} rk(i) + wn(i) + \mu [x(i) - f(k(i), n(i))],$$

where  $\mu \geq 0$  is the multiplier. The first order conditions for this problem are:

$$r = \mu f_k \quad (5)$$

$$w = \mu f_n \quad (6)$$

$$x = f.$$

Here,  $\mu, k$  and  $n$  are to be determined as functions of  $x, r, w$ . To obtain an expression for  $\mu$ , solve (5) for  $k/n$ :

$$\frac{k}{n} = f_k^{-1} \left( \frac{r}{\mu} \right).$$

Substitute this into (6) and obtain:

$$w = \mu f_n \left[ f_k^{-1} \left( \frac{r}{\mu} \right) \right].$$

Solving this for  $\mu$  and taking into account our functional form assumption, we find:

$$\mu = \mu(r, w) = \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{1}{\alpha}\right)^\alpha (r)^\alpha (w)^{1-\alpha}.$$

To interpret  $\mu$ , note that it is the derivative of  $C$  with respect to  $x(i)$  in the Lagrangian expression of the firm minimization problem. Thus,  $\mu(r, w)$  is the marginal cost of producing  $x(i)$ . Another intuitive way to see this interpretation of  $\mu$  is to consider (5), which can be rewritten,  $\mu = r/f_k$ . Note that  $r$  is the change in cost with respect to a unit change in capital, while  $f_k$  is a change in output with respect to a unit change in capital, so that the ratio is the change in cost with respect to a change in output. That is,

$$\mu = \frac{r}{f_k} = \frac{\frac{dCost}{dk}}{\frac{dx}{dk}} = \frac{dCost}{dx} \times \frac{dk}{dk} = \frac{dCost}{dx}.$$

This establishes that the marginal cost of producing  $x(i)$  is independent of the value of  $x(i)$ . (The linear homogeneity of the production function is what guarantees this.) From this observation, together with the fact that  $C(0, r, w) = 0$ , we conclude that the cost function of the  $i^{th}$  intermediate good firm has the following representation:

$$C(x(i), r, w) = \mu(r, w) \times x(i).$$

Taking into account that the intermediate good firm must respect its demand curve, the profit maximization problem is:

$$\pi(i) = \max_{x(i)} P(x(i), y)x(i) - \mu(r, w)x(i).$$

The first order condition equates marginal revenue,  $P'(x(i), y)x(i) + P(x(i), y)$ , to marginal cost,  $\mu(r, w)$ . Taking into account our functional form assumptions,

$$\lambda \left(\frac{y}{x(i)}\right)^{1-\lambda} = \mu(r, w).$$

Note that the value of  $x(i)$  that solves this is independent of  $i$ . Similarly, the choice of  $p(i)$  is independent of  $i$ . These observations are not surprising in view of the symmetry of the intermediate good firm

problems. Denote the optimal values of  $x(i)$  and  $p(i)$  by  $x$  and  $p$ , respectively. From the fact that (2) must be satisfied in equilibrium, we have that in equilibrium,  $x = y$ , so that, using the previous expression and the demand curve:

$$\begin{aligned}\lambda &= \mu(r, w) \\ p &= 1.\end{aligned}$$

That is, in equilibrium the marginal cost is equated to  $\lambda$ , the parameter in the final good production function. In addition, the price of the intermediate input is unity. It is interesting to see the value of profits in equilibrium:

$$\begin{aligned}\pi &= P(x, y)x - \mu(r, w)x \\ &= x[1 - \lambda].\end{aligned}$$

Note that when  $\lambda = 1$ , then profits are zero. This reflects that, in this case, the intermediate inputs are perfect substitutes so that the ‘monopolist’ does not have any exploitable monopoly power. In this case, marginal revenue is just the price of the intermediate good and  $P' = 0$ . For lower values of  $\lambda$ , each intermediate good firm becomes less substitutable with the others, and the monopolist has more monopoly power. As a result, profits per unit of sales,  $\pi/x$ , rises. A crucial thing to note is that, according to (5)-(6), the monopolist pays factors of production less than their marginal products:

$$\begin{aligned}r &= \lambda f_k < f_k \\ w &= \lambda f_n < f_n.\end{aligned}\tag{7}$$

The equations that summarize the firm sector are these, together with (2) which reduces to

$$y = k^\alpha n^{1-\alpha},$$

given that  $y = x(i)$  for all  $i$ . These equations are conditional on the values of  $k$  and  $n$ , which we have taken as given in our discussion of the firm sector. Notice that these three equations look exactly like the equations corresponding to the firm sector in our competitive decentralization of the neoclassical model, with the exception of the appearance of  $\lambda$  in the firm first order conditions.

To determine  $k$ , and  $n$ , we have to bring in the households. Their problem is to solve, at each date,  $t$  :

$$\max \sum_{j=0}^{\infty} \beta^j u(c_{t+j}),$$

subject to

$$c_{t+j} + k_{t+1+j} - (1 - \delta)k_{t+j} \leq r_{t+j}k_{t+j} + w_{t+j}n_{t+j} + \pi_{t+j} + \int_0^1 \pi_{t+j}(i)di,$$

and  $n \leq 1$ . Note that households are treated as receiving profits from all firms, final and intermediate. The household optimally chooses  $n = 1$  and its first order condition for capital is:

$$u'(c_{t+j}) = \beta u'(c_{t+1+j}) [r_{t+1+j} + 1 - \delta], \quad j = 0, 1, 2, \dots$$

Substituting out for the equilibrium value of the rental rate of capital,

$$u'(c_{t+j}) = \beta u'(c_{t+1+j}) [\lambda f_{k,t+j+1} + 1 - \delta], \quad j = 0, 1, 2, \dots$$

Notice that this first order condition resembles the first order condition of the efficient allocations in the neoclassical model, with a crucial exception. Instead of  $f_k$  appearing here, it is something less. A consequence of this is that when households make their decisions about saving, they treat the payoff from capital as  $r + 1 - \delta$ , which is less than the actual payoff,  $f_k + 1 - \delta$ . Because, in effect, the equilibrium offers households insufficient incentive to save, it will produce a socially inefficiently low level of capital.

A sequence of markets equilibrium for this economy is a sequence of prices,  $\{p_t(i), r_t, w_t\}_{t=0}^{\infty}$ , and quantities,  $\{n_t(i), k_t(i), n_t, k_{t+1}, c_t\}_{t=0}^{\infty}$ , such that the household and firm problems are satisfied given the prices and the quantities. In addition, we require market clearing,  $\int n_t(i)di = n_t$ ,  $\int k_t(i)di = k_t$ , for each  $t$ .

- (a) Use the linear homogeneity of the production function, and expressions for profits to show that, in equilibrium,

$$\begin{aligned} & r_t k_t + w_t n_t + \pi_t + \int_0^1 \pi_t(i)di \\ &= f(k_t, n_t), \end{aligned}$$

so that the resource constraint is satisfied.

(b) Show that, in a steady state,

$$k = \left[ \frac{\lambda\alpha}{\frac{1}{\beta} - (1 - \delta)} \right]^{\frac{1}{1-\alpha}}.$$

(c) Show that the efficient allocations of the economy with monopolists solve:

$$\max_{k_{t+1}, n_t} \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to

$$c_t + k_{t+1} - (1 - \delta)k_t \leq f(k_t, n_t).$$

(Hint: note that I dropped the whole thing about (2). In effect, you need to show that this not a binding restriction on the problem of determining the efficient allocations.) Thus, the efficient allocations in the monopoly economy coincide with those of the neoclassical economy. However, because of the presence of monopoly power, the equilibrium allocations are not efficient. This is consistent with the result for the steady state capital stock in (b) above, which establishes that when  $\lambda < 1$ , the equilibrium steady state capital stock in the economy is less than the efficient level of capital, which we derived in class.

(d) Modify the household budget constraint in the following way:

$$c_{t+j} + k_{t+1+j} - (1 - \delta)k_{t+j} \leq (1 + \tau_{t+j})r_{t+j}k_{t+j} + w_{t+j}n_{t+j} + \pi_{t+j} + \int_0^1 \pi_{t+j}(i)di - T_{t+j}$$

Here,  $\tau_{t+j}$  is a tax subsidy on the household's capital income. In addition,  $T_{t+j}$  is a lump sum transfer from the household to the government. By 'lump sum' I mean that the magnitude of the transfer is independent of any choice by the household. We require that the government balance its books in each period:

$$\tau_t r_t k_t = T_t,$$

where  $k_t$  is the economy-wide average stock of capital, and *not* the quantity of capital chosen by any particular household (otherwise, we could not maintain our assumption that from the perspective

of the individual household,  $T_t$  is lump sum). Show that there is a value of  $\tau_t$ ,  $t = 0, 1, 2, \dots$  such that the allocations in a sequence of market equilibrium are efficient. Note that this value is positive. This reflects that, to steer households towards the efficient level of investment, they need an additional incentive beyond what the market gives them. The market systematically gives them too little incentive because of the presence of monopoly power. In thinking about whether a tax policy can be found which selects the efficient allocations, be sure to also think about the allocations in date 0.