

Homework #6
Economics 411
Due Wednesday, November 14
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1. (Vintage interpretation of exogenous, embodied technical change model.) Consider an economy with capital of different vintages. At time t , the amount of capital of vintage τ , $k_{t,\tau}$, $\tau = 1, 2, 3, \dots$, is

$$k_{t,\tau} = \gamma^{t-\tau}(1 - \delta)^{\tau-1}i_{t-\tau},$$

where $\gamma > 1$, $0 < \delta < 1$, $i_{t-\tau}$ is the amount of investment, in time $t - \tau$ consumption units, applied in period $t - \tau$. Capital which has vintage τ in period t has vintage $\tau + 1$ in period $t + 1$. Investment expenditures at time t , i_t , must all be applied to the latest vintage (for a model in which investment in old vintages is feasible and desirable, see Chari and Hopenhayn, JPE, 1991) and results in $k_{t+1,1} = \gamma^t i_t$ units of new-vintage period $t + 1$ installed capital goods. Consider a given amount of investment, i . Note that this investment applied in period $t + 1$ produces more new-vintage installed capital (i.e., $\gamma^{t+1}i$) than the same level of investment applied in period t (i.e., $\gamma^t i$). This reflects the assumption, $\gamma > 1$ which is designed to capture the notion that there is exogenous technical progress that is embodied in new capital equipment. Note that the efficiency of a particular vintage stays constant over time, it's just that the efficiency of each succeeding vintage is greater than the efficiency of the previous one.

Capital of each vintage is operated with labor to produce a homogeneous output good, $y_{t,\tau}$ according to the following production function:

$$y_{t,\tau} = k_{t,\tau}^\alpha n_{t,\tau}^{1-\alpha}, \quad 0 < \alpha < 1, \quad \tau = 1, 2, 3, \dots$$

Suppose there is a competitive market in capital of different vintages and in labor. Each vintage of capital has the same rental rate, r_t , since capital is measured in common efficiency units. Similarly, the wage rate is w_t .

- (a) Show that a firm's profit maximizing choice of $n_{t,\tau}$ gives rise to the following relationships:

$$y_t = k_t^\alpha n_t^{1-\alpha}, \quad (1 - \alpha) \left(\frac{k_t}{n_t} \right)^\alpha = w_t, \quad \alpha \left(\frac{k_t}{n_t} \right)^{\alpha-1} = r_t,$$

where

$$y_t = \sum_{\tau=1}^{\infty} y_{t,\tau}, \quad k_t = \sum_{\tau=1}^{\infty} k_{t,\tau}, \quad n_t = \sum_{\tau=1}^{\infty} n_{t,\tau}.$$

- (b) Show that 'aggregate capital', k_t , evolves as in the Sollow II model:

$$k_{t+1} = (1 - \delta)k_t + \gamma^t i_t.$$

2. Consider a two-sector economy with the following preferences:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t),$$

where

$$u(c, l) = \begin{cases} \frac{[c(1-l)^\eta]^{1-\gamma}}{1-\gamma}, & \gamma \neq 1, \gamma > 0 \\ \log c + \eta \log(1-l), & \gamma = 1 \end{cases}.$$

Consumption goods are produced using the following technology:

$$c_t \leq A k_{c,t}^\alpha l_t^{(1-\alpha)}$$

and investment goods are produced using the following technology:

$$I_t = b k_{I,t},$$

where

$$\begin{aligned} k_{t+1} &= (1 - \delta) k_t + I_t \\ k_t &= k_{c,t} + k_{I,t}, \quad k_{c,t}, k_{I,t} \geq 0, \end{aligned}$$

where $\delta \in (0, 1)$ and $b > \delta$. Also, the initial stock of capital, $k_0 > 0$, is given. The following condition will later be useful to guarantee boundedness of utility:

$$\beta (1 - \delta + b)^{\alpha(1-\gamma)} < 1.$$

- (a) Show that the efficient allocations solve the following problem:

$$V(k_0) = \max_{k_{t+1}, l_t \in \Gamma(k_t)} \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1}, l_t).$$

Display the function, F , and the correspondence, Γ .

- (b) Show that V has the following form:

$$V(k_0) = k_0^{\alpha(1-\gamma)} w,$$

where w is finite. Establish the finiteness of w for $\gamma > 1$ and $\gamma < 1$.

- (c) Show that w is the fixed point of a particular functional equation. Does the functional equation satisfy Blackwell's conditions to be a contraction? Display formulas which can be used to solve for w as well as the optimal level of employment and the optimal growth rate of capital.
- (d) Compute the price of capital, $P_{k',t}$, in this model. Show that along a growth path, $P_{k',t} \rightarrow 0$. Show that the marginal product of capital also converges to zero. Show that the rate of return on capital is constant.
- (e) Does the economy satisfy the convergence property?

3. The Ak model leads one to consider the following functional equation:

$$T(v) = \max_{0 \leq \lambda \leq A+1-\delta} \frac{[A+1-\delta-\lambda]^{(1-\sigma)}}{1-\sigma} + \beta\lambda^{(1-\sigma)}v.$$

Suppose $\sigma > 1$, $A > \delta$, and $\beta(A+1-\delta)^{1-\sigma} < 1$.

- (a) Show: $T(v) = \infty$ for $v > 0$, $T(0) = \frac{[A+1-\delta]^{(1-\sigma)}}{1-\sigma}$.
- (b) Show: the derivative of T at $v = v_0 < 0$ is:

$$\frac{dT(v_0)}{dv} = \beta\lambda(v_0)^{(1-\sigma)},$$

where

$$\lambda(v_0) = \operatorname{argmax}_{0 \leq \lambda \leq A+1-\delta} \frac{[A+1-\delta-\lambda]^{(1-\sigma)}}{1-\sigma} + \beta\lambda^{(1-\sigma)}v_0.$$

- (c) Show that T does not satisfy the conditions of Blackwell's theorem.
- (d) What happens to $\lambda(v)$ as $v \rightarrow -\infty$?
- (e) What does the graph of $T(v)$ versus v for $v \leq 0$ look like? Does it cross a 45° line drawn in the negative orthant? Draw the $T(v)$ function by hand, emphasizing its qualitative features.
- (f) Explain, using the graph you just developed, why $T^j v_0 = v^*$ as $j \rightarrow \infty$, for every $v_0 < 0$. Also, explain why v^* such that $v^* = T(v^*)$, is unique.