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Econ 411-1, Fall 2007

MIDTERM EXAM

There are six questions. The total number of possible points is 100, and the number of points per question is given in parentheses. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have until 10:50am. Good luck!

Following is the Stokey-Lucas canonical model:

$$\max_{x_{t+1}\in\Gamma(x_t)} \sum_{t=0}^{\infty} \beta^t F\left(x_t, x_{t+1}\right),$$

where Γ is a correspondence which associates with each $x_t \in X$ a set $\Gamma\left(x_t\right) \in X$ of feasible choices for x_{t+1}. Also, F is a function, $F : A \rightarrow R$, where

$$A = \left\{(x, y) : x \in X,\ y \in \Gamma(x)\right\}.$$

Following are some assumptions that we have used:

$$\begin{aligned} & X \text{ is a convex subset of } R_+^l; \qquad (1) \\ \Gamma \ &: \ X \rightarrow X,\ \Gamma \text{ non-empty, compact and continuous} \end{aligned}$$

$$F : A \rightarrow R,\ \text{bounded, continuous, } 0 < \beta < 1. \qquad (2)$$

$$\text{for each fixed } y,\ F(x,y) \ \text{ is strictly increasing} \qquad (3)$$

$$\Gamma \text{ is monotone,} \qquad (4)$$

$$F \text{ is strictly concave} \qquad (5)$$

$$\Gamma \text{ is convex} \qquad (6)$$

$$F \text{ is continuously differentiable on the interior of } A \qquad (7)$$

1. (20) Shorter questions:

   (a) (6) Explain carefully what it means that the correspondence, Γ, is convex.

(b) (6) Setting up a competitive equilibrium is problematic if firms have strictly concave production functions and rent factors in competitive factor markets. Explain why this is so (you may assume a production function in a single input).

(c) (8) Consider the resource constraint in the neoclassical model:

$$c_t + i_t \leq f(k_t),$$

where $c_t \geq 0$, f is strictly concave, $f(0) = 0$, $f'(k) \to 0$ as $k \to \infty$ and $f'(k) \to \infty$ as $k \to 0$. Also,

$$k_{t+1} = (1 - \delta) k_t + i_t.$$

Define the constraint set, Γ, for this economy. Explain the meaning of assumption (4). Is this assumption satisfied if we impose the constraint, $i_t \geq 0$? What if we impose the constraint, $k_{t+1} \geq 0$? Explain.

2. (25) Let $x_1^*,\ x_2^*,\ x_3^*, \ldots$ solve the sequence representation of the SL canonical problem. Suppose $\{x_t^*\}$ is interior, that is, $x_t^* \in int(X)$, $x_{t+1}^* \in int(\Gamma(x_t^*))$ for $t = 0, 1, \ldots$. Also, suppose $A4.3$, $A4.7$ and $A4.9$ are satisfied and, for simplicity, $l = 1$. Then, the following must be true:

$$F_2(x_t^*, x_{t+1}^*) + \beta F_1(x_{t+1}^*, x_{t+2}^*) = 0,\ t = 0, 1, 2, \ldots.$$

The purpose of this question is to get you to prove this proposition.

(a) (10) Consider a particular class of variations on the optimal plan: $x_1^*,\ x_2^* + \delta,\ x_3^*, \ldots$, where $|\delta| < \epsilon$ and $\epsilon > 0$. Argue that, for ϵ small enough, every element in this class of variations is feasible. To establish the importance of continuity of Γ, illustrate how this statement need not be true when Γ fails to be continuous (you can use a non-continuous version of the one-dimensional feasibility correspondence in the neoclassical model).

(b) (10) Let $S : D \to R$, where $D = \{\delta : |\delta| < \epsilon\}$ and

$$S(\delta) = F(x_0^*, x_1^*) + \beta F(x_1^*, x_2^* + \delta) + \beta^2 F(x_2^* + \delta, x_3^*) + \sum_{t=3}^{\infty} \beta^t F(x_t^*, x_{t+1}^*).$$

Prove formally that S is a strictly concave function some $\epsilon > 0$.

(c) (5) Prove the proposition.

3. (20) Consider a household which solves the following problem:

$$v(k, r, w) = \max_{c,l \in B(k,r,w)} u(c, l),$$

where $u : \Re^2_+ \to \Re$ is a strictly concave, twice continuously differentiable, strictly increasing function in its two arguments: consumption, c, and leisure, l. The constraints the household must obey in selecting c, l are summarized by B :

$$B(k, r, w) = \{c, l : 0 \leq c \leq rk + w(1 - l),\ 0 \leq l \leq 1\}.$$

Here, $r > 0$ is the market rental rate on capital and $w > 0$ is the market wage rate, neither of which the household can control. Also, $k > 0$ is the household's stock of capital. Prove that the derivative of v with respect to k exists, and display a formula for it. If you make use of a theorem to help prove your result, be sure to state it clearly.

4. (20) Consider the following operator on the space of functions:

$$T[w](x) = \max_{x' \in \Gamma(x)} F(x, x') + \beta w(x'),$$

where $x \in X \subset R^l$, $\Gamma : X \to X$, $F : A \to R$, $A = \{(x, y) : x \in X, y \in \Gamma(x)\}$. Here, X is convex; Γ is non-empty, compact and continuous; F is bounded and continuous. Suppose assumptions (5) and (6) are satisfied. Prove that if w is continuous, bounded and weakly concave, then $T[w]$ is strictly concave. Be sure to make very clear what role the assumptions play in establishing the result.

5. (15) Consider the following two-sector model of optimal growth. A planner seeks to maximize the utility of the representative agent given by $\sum_{t=0}^{\infty} \beta^t u(c_t)$, where c_t is consumption at t. Sector 1 produces consumption goods using capital, k_{1t}, and labor, n_{1t}, according to the production function, $c_t \leq f_1(k_{1t}, n_{1t})$. Sector 2 produces the capital good according to the production function $k_{t+1} \leq f_2(k_{2t}, n_{2t})$. The constraint on labor is $n_{1t} + n_{2t} = 1$, where 1 denotes the total amount of labor supplied. The other constraints include $n_{it}, k_{it} \geq 0$, $i = 1, 2$, and

$k_{t+1} \geq 0$. The sum of the amounts of capital used in each sector cannot exceed the initial capital in the economy, that is, $k_{1t} + k_{2t} \leq k_t$, and $k_0 > 0$, given. Show how this model can be set up in the Stokey-Lucas canonical form. (Hint: you need to define the objects, F, A, Γ, X.)