Christiano
411, Fall, 2008

FINAL EXAM

Each of the four questions is worth 25 points. Allocate your time accordingly. If a question seems ambiguous, state why, sharpen it up and answer the revised question. Good luck!

1. Consider the sequence representation of the following two-sector planning problem:

$$\max_{\{c_t, i_t, k_{1,t+1}, k_{2,t+1}, l_{1,t}, l_{2,t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{array}{c} c_t \leq z_t F(k_{1,t}, l_{1,t}) \\ i_t \leq q_t z_t F(k_{2t}, l_{2t}) \end{array}.$$

Here, $k_{it}$ and $l_{it}$ are capital and labor allocated to sector $i$, $i = 1, 2$. Assume that factors can be freely moved between sectors, subject to:

$$k_t \equiv k_{1,t} + k_{2,t}, \;\; l_{1,t} + l_{2,t} = l,$$

where $k_t$ is the aggregate stock of capital given at the beginning of time $t$ and $l$ is the (fixed) amount of labor effort supplied by households. Finally, we also require

$$k_{t+1}, c_t \geq 0, k_0 > 0$$

and the identity

$$k_{t+1} \leq (1-\delta) k_t + i_t.$$

The sequences $\{z_t, q_t\}_{t=0}^{\infty}$ are exogenously given. It is assumed that $u$ is continuously differentiable, strictly increasing, and strictly concave, that $F$ is continuously differentiable, strictly increasing in both arguments, homogeneous of degree one, and strictly quasiconcave, and that $\delta, \beta \in (0, 1)$.

(a) Show that a necessary condition for optimization is $\frac{k_{1t}}{l_{1t}} = \frac{k_{2t}}{l_{2t}}$, and that this implies the constraint set above can be replaced by

$$\begin{aligned} c_t + \tfrac{i_t}{q_t} &\leq z_t F(k_t, l) \\ k_{t+1} &= (1-\delta)k_t + i_t \text{ and } k_0 > 0 \\ c_t, k_{t+1} &> 0. \end{aligned}$$

(b) Assume that the two productivity change series follow:

$$z_t = \gamma_z^t \text{ and } q_t = \gamma_q^t,$$

where $\gamma_z \neq \gamma_q$ are each greater than one. Suppose $F$ has a Cobb-Douglas form, and $u(c) = c^{1-\sigma}/(1-\sigma)$, $\sigma < 1$. Let a *steady-state growth path* be a situation in which $c_t$, $k_t$, and $i_t$ are growing at a constant rate. Let $\gamma_k$ and $\gamma_c$ denote the steady-state growth rates of the capital stock and consumption, respectively. Develop equations that determine $\gamma_k$ and $\gamma_c$ as a function of the parameters of the problem.

(c) Scale the variables and show that the sequence representation of the planning problem can be expressed as a problem involving no growing variables.

(d) Express the non-growing, scaled, representation of the planning problem in functional equation form.

(e) Write the competitive equilibrium that corresponds to the scaled planning problem as a sequence of markets competitive equilibrium.

(f) Write the competitive equilibrium in recursive competitive equilibrium form.

2. Consider an economy in which there are two types of households, capitalists and workers. Capitalists accumulate capital, but have no labor power. Workers supply labor, but have no access to capital markets. There are two periods. In the first period, capitalists have an endowment, $\omega$, which they can accumulate in the form of capital, $k$, or they can consume, $c_1$. Their first period budget constraint is:

$$c_1 + k \leq \omega.$$

Their second period budget constraint is:

$$c_2^k \leq R(1-\delta)k,$$

where $R$ is the rental rate of capital (an exogenous parameter of the model) and $\delta$ is the capital income tax rate. The lifetime utility of capitalists is:

$$u^k(c_1, c_2^k) = c_1 + c_2^k$$

We assume that if the capitalist is indifferent between consuming in periods 1 or 2, then they choose to do all their consumption in period 2.

Consider the workers. In period 1, they have no utility. In period 2, their budget constraint is:

$$c_2^w \leq (1-\tau)l,$$

where $\tau$ is the labor tax rate and the wage rate is set to unity. Workers' utility function is

$$u^w(c_2^w, l) = c_2^w - \frac{1}{2}l^2.$$

The social welfare function in this society is:

$$u(u^w, u^k) = u^w + u^k.$$

Suppose the government faces an exogenously determined required level of spending, $g$, where

$$(R-1)\omega < g < (R-1)\omega + \frac{1}{4}$$

The government's budget constraint is:

$$g \leq \tau l + \delta R k.$$

(a) Consider the best equilibrium, relative to the given social welfare function. Explain carefully why the labor tax rate in the best equilibrium must be positive. Call the policies in the best equilibrium, the Ramsey policies.

(b) Suppose the task of administering government policy is given to an administrator who is benevolent in the sense that he is interested in maximizing the social welfare function. Suppose this person must impose the policy during an 'administration period', which occurs at the beginning of period 2, before the labor decision has been taken. Prove that this administrator will deviate from the Ramsey policies. Provide intuition.

(c) Suppose everyone understands that policy will be implemented by the benevolent administrator in (b). Define a sustainable equilibrium for this economy, and explain the outcomes that occur in sustainable equilibrium.

(d) Suppose the task of administering government policy in the administration period is given to an administrator who is *not* benevolent. The administrator has preferences:

$$u(u^w, u^k; \lambda) = u^w + \lambda u^k.$$

An administrator with preferences $\lambda > 1$ is partial towards capitalists. Explain why it is that there is a $\lambda > 1$, such that an administrator with this value of $\lambda$ would choose, in the administration period, not to deviate from the Ramsey policies.

(e) Given a choice between the sustainable equilibrium outcomes described in (c) and the outcomes in (d), is it possible that workers might prefer an administrator that is partial to capitalists, if they were asked at the beginning of period 1? Explain.

3. Consider an economy in which the representative household has the following preferences:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t), \ 0 < \beta < 1,$$

where $u$ satisfies the usual restrictions. The resource constraint is:

$$c_t + k_{t+1} - (1-\delta)k_t \leq y_t,$$

and $0 < \delta < 1$. Final goods, $y_t$, are produced using the linear homogeneous technology:

$$y_t = \left[ \int_0^1 x_t(i)^{\lambda} di \right]^{\frac{1}{\lambda}}, \ \lambda > 1,$$

where $x_t(i)$ is the quantity of the $i^{th}$ intermediate good used. The technology for producing intermediate goods is

$$x_t(i) = k_t(i)^{\mu} n_t(i)^{\gamma}, \ \mu, \gamma > 0, \ 1 < \mu + \gamma \leq \psi \text{ for all } i \in (0, \infty).$$

Here, $\psi$ is a parameter to be discussed below.

(a) Decentralize this economy and define a symmetric, sequence of markets equilibrium. In the equilibrium, give the final good technology to a representative competitive firm; give the technology for producing the $i^{th}$ intermediate good to a monopolist; and suppose the representative household rents capital and labor in homogeneous, competitive markets. Only consider symmetric equilibria, in which all intermediate good firms behave identically. Let $p_t(i)$ denote the price of the $i^{th}$ intermediate good and let $w_t$ and $r_t$ denote the wage rate and capital rental rate, respectively. All date $t$ prices are denoted in units of the date $t$ consumption good.

(b) Derive an expression for the demand curve faced by the monopolist. What happens to the slope of the demand curve (with $p_t(i)$ on the vertical axis and $x_t(i)$ on the horizontal) as $\lambda \to 1$? Provide intuition.

(c) What restriction on $\psi$ is necessary if the monopolist is to have a well-defined maximization problem?

(d) Derive an expression for the markup (*i.e.*, the ratio of price to marginal cost) charged by the typical monopolist.

(e) Derive a simple expression for the level of profits in a symmetric equilibrium.

(f) Show that a necessary condition for allocations to constitute an interior equilibrium, is that they satisfy:

$$u_{c,t} = \beta u_{c,t+1} \left[ \zeta \frac{y_{t+1}}{k_{t+1}} + 1 - \delta \right], \quad \frac{-u_{n,t}}{u_{c,t}} = \xi \frac{y_t}{n_t}, \ t = 0, 1, 2, \ldots ,$$

where $u_{c,t}$ and $u_{n,t}$ are the derivatives of $u$ with respect to the first and second arguments. Derive expressions relating $\zeta$ and $\xi$ to the model parameters.

4. Suppose that entrepreneurs possess the following technology for converting capital, $k$, into output:

$$y(\omega) = \omega k^{\alpha},\ 0 < \alpha \leq 1,$$

where $\omega$ is a technology shock drawn independently by the entrepreneur from a distribution with $E\omega = 1$ and cumulative distribution function $F(x) \equiv prob[\omega \leq x]$. The realization of $\omega$ is observed by the entrepreneur, and can be seen by a lender only by paying a monitoring cost, $\mu k^{\alpha}$. Entrepreneurs sell their output in a competitive market at a price of unity. Entrepreneurs have their own net worth, $n$, to use in purchasing capital, and suppose there are many entrepreneurs with each possible level of $n$. Consider an entrepreneur with net worth, $n$, who purchases an amount of capital, $k > n$. The entrepreneur borrows $b \equiv k - n$ at gross rate of interest, $Z$, from a bank. There is a large number of banks that specialize in lending to entrepreneurs with each level of net worth, $n$, and there is free entry into banking. In case the entrepreneur's revenue, $y(\omega)$, falls below the required payment to the bank, $Z(k-n)$, the entrepreneur declares bankruptcy and is monitored. In addition, the bank takes whatever the entrepreneur has. Let $\bar{\omega}$ be defined by

$$\bar{\omega} k^{\alpha} = Z(k-n).$$

All banks have access to a competitive market in which they can borrow as much or as little as they want, at gross rate of interest, $R$. Entrepreneurial utility prior to production is proportional to their expected revenues net of bank costs.

(a) Show that the expected profits of an entrepreneur with net worth $n$, interest rate $Z$, and loan amount $k - n$ can be written

$$[1 - \Gamma(\bar{\omega})] k^{\alpha},$$

where

$$\Gamma(\bar{\omega}) = [1 - F(\bar{\omega})]\bar{\omega} + \int_0^{\bar{\omega}} \omega dF(\omega).$$

(b) Suppose that each bank is diversified in terms of their customers. Show that the average, across all its customers, of the revenues of a bank which lends to entrepreneurs with net worth $n$ is

$$[\Gamma(\bar{\omega}) - \mu F(\bar{\omega})] k^{\alpha}.$$

(c) Display the zero profit condition for the banks that lend to entrepreneurs with net worth, $n$. Write this in terms of the variables, $b$, $R$, $Z$ and $n$ only (not $\bar{\omega}$ or $k$, please). Explain why (outside of very special model parameter values) there cannot be an equilibrium in which banks offer an interest rate $Z$ and banks allow entrepreneurs to borrow as much as they want at that interest rate. (You may assume without proof that for each $Z$ there is a unique $b$ that satisfies the zero profit condition.)

(d) Let the bank zero profit condition define a menu of contracts, $(b, Z)$, that banks who lend to entrepreneurs with net worth $n$ offer in equilibrium. Display a constrained optimization problem that characterizes which $b$, $Z$ combination entrepreneurs with net worth, $n$, select from this menu. You may assume that the chosen contract is interior and is characterized by the first order conditions being satisfied as a strict equality. Will the interest rate in the selected contract vary with entrepreneurial net worth, $n$? Does your answer to the last question change across the cases, $\alpha = 1$ and $\alpha < 1$? Establish your answer carefully.