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FINAL EXAM

Allocate your time to the following four questions in proportion to the number of points available. If a question seems ambiguous, state why, sharpen it up and answer the revised question. Good luck!

1. Suppose that entrepreneurs possess the following technology for converting capital, k, into output:

$$y(\omega) = \omega k^{\alpha},\ 0 < \alpha \leq 1,$$

where ω is a technology shock drawn independently by the entrepreneur from a distribution with $E\omega = 1$ and cumulative distribution function $F(x) \equiv prob[\omega \leq x]$. The realization of ω is observed by the entrepreneur, and can be seen by a lender only by paying a monitoring cost, μk^{α}. Entrepreneurs sell their output in a competitive market at a price of unity. Entrepreneurs have their own net worth, n, to use in purchasing capital, and suppose there are many entrepreneurs with each possible level of n. Consider an entrepreneur with net worth, n, who purchases an amount of capital, $k > n$. The entrepreneur borrows $k - n$ at gross rate of interest, Z, from a bank. There is a large number of banks that specialize in lending to entrepreneurs with each level of net worth, n, and there is free entry into banking. In case the entrepreneur's revenue, $y(\omega)$, falls below the required payment to the bank, $Z(k-n)$, the entrepreneur declares bankruptcy and is monitored. In addition, the bank takes whatever the entrepreneur has. Let $\bar{\omega}$ be defined by

$$\bar{\omega} k^{\alpha} = Z(n-k).$$

All banks have access to a competitive market in which they can borrow as much or as little as they want, at gross rate of interest, R. Entrepreneurial utility prior to production is proportional to their expected revenues net of bank costs.

(a) Show that the expected profits of an entrepreneur with net worth n, interest rate R, and loan amount $n-k$ can be written

$$[1-\Gamma(\bar{\omega})]\,k^{\alpha},$$

where

$$\Gamma(\bar{\omega}) = [1-F(\bar{\omega})]\,\bar{\omega} + \int_0^{\bar{\omega}} \omega dF(\omega).$$

ans:

$$\begin{aligned}
&\overbrace{\int_{\bar{\omega}}^{\infty} \omega dF(\omega)\, k^{\alpha}}^{\text{revenues outside of bankruptcy}} \quad - \overbrace{[1-F(\bar{\omega})]\,Z(n-k)}^{\text{costs outside of bankruptcy}} \\
&\left[\int_{\bar{\omega}}^{\infty} \omega dF(\omega) - [1-F(\bar{\omega})]\,\bar{\omega}\right] k^{\alpha} \\
&\left[1-\int_0^{\bar{\omega}} \omega dF(\omega) - [1-F(\bar{\omega})]\,\bar{\omega}\right] k^{\alpha} \\
&\left[1-\left(\int_0^{\bar{\omega}} \omega dF(\omega) + [1-F(\bar{\omega})]\,\bar{\omega}\right)\right] k^{\alpha} \\
= \;&[1-\Gamma(\bar{\omega})]\,k^{\alpha}
\end{aligned}$$

(b) Show that the revenues of banks who lend to entrepreneurs with net worth, n, can be expressed as follows:

$$[\Gamma(\bar{\omega}) - \mu F(\bar{\omega})]\,k^{\alpha}.$$

ans:

$$\begin{aligned}
&\overbrace{[1-F(\bar{\omega})]\,\bar{\omega}k^{\alpha}}^{\text{revenues from non-bankrupt entrepreneurs}} \quad + \quad \overbrace{\int_0^{\bar{\omega}} \omega dF(\omega)\, k^{\alpha}}^{\text{revenues from bankrupt entrepreneurs}} \quad - \quad \overbrace{\mu k^{\alpha} F(\bar{\omega})}^{\text{monitoring costs}} \\
= \;&[\Gamma(\bar{\omega}) - \mu F(\bar{\omega})]\,k^{\alpha}.
\end{aligned}$$

(c) Derive the zero profit condition of banks in the market for lending to entrepreneurs with net worth, n. Write this in terms of the variables, b, R, Z and n only (no $\bar{\omega}$ or k, please). Explain why (outside of very special model parameter values) there cannot be an equilibrium in which banks offer an interest rate Z and they allow entrepreneurs to borrow as much as they want at that interest rate. (You may assume without proof that for each Z there is a unique b that satisfies the zero profit condition.)

ans: here is the zero profit condition. It is obtained simply by equating banks' expected revenues to Rb, and substituting out for k in terms of $b+n$ and substituting out for $\bar{\omega}$ in terms of Z and b:

$$\left[\Gamma\left(\frac{Zb}{(b+n)^{\alpha}}\right)-\mu F\left(\frac{Zb}{(b+n)^{\alpha}}\right)\right](b+n)^{\alpha}=Rb$$

Assume that for each Z there is exactly one b that satisfies this relation. Entrepreneurs who choose b to maximize utility will choose the b that satisfies this relation only for a measure zero set of parameter values. The only Z, b combinations that can be offered in equilibrium are the ones that generate zero profits, and these are the ones given by the above equation.

(d) Let the bank zero profit condition define a menu of contracts, (b, Z), that banks who lend to entrepreneurs with net worth n offer in equilibrium. Display a constrained optimization problem that characterizes which b, Z combination entrepreneurs with net worth, n, select from this menu. You may assume that the chosen contract is interior and is characterized by the first order conditions being satisfied as a strict equality. Will the interest rate in the selected contract vary with entrepreneurial net worth, n? Does your answer to the last question vary with the value of α?

ans: the problem solved by entrepreneurs with net worth n, when choosing a b, Z combination in the menu offered is the following:

$$\max_{\bar{\omega},b}\left[1-\Gamma(\bar{\omega})\right](b+n)^{\alpha}+\lambda\left\{\left[\Gamma(\bar{\omega})-\mu F(\bar{\omega})\right](b+n)^{\alpha}-Rb\right\}$$

the first order conditions are:

$$\begin{aligned}
b &: \quad \alpha\left[1-\Gamma(\bar{\omega})\right](b+n)^{\alpha-1}+\lambda\left\{\alpha(b+n)^{\alpha-1}\left[\Gamma(\bar{\omega})-\mu F(\bar{\omega})\right]-R\right\}=0 \\
\bar{\omega} &: \quad \lambda=\frac{\Gamma'(\bar{\omega})}{\Gamma'(\bar{\omega})-\mu F'(\bar{\omega})}.
\end{aligned}$$

Substituting out for λ from the second relation into the first, we obtain the following condition:

$$\alpha\left[1-\Gamma\left(\bar{\omega}\right)\right]\left(b+n\right)^{\alpha-1}+\frac{\Gamma'\left(\bar{\omega}\right)}{\Gamma'\left(\bar{\omega}\right)-\mu F'\left(\bar{\omega}\right)}\left\{\alpha\left(b+n\right)^{\alpha-1}\left[\Gamma\left(\bar{\omega}\right)-\mu F\left(\bar{\omega}\right)\right]-R\right\}=0$$

Note that when $\alpha = 1$, then n disappears from this equation and the value of $\bar{\omega}$ that solves this equation is the same for each n. If $\alpha < 1$, then entrepreneurs with different n obtain loan contracts with different rates of interest.