

Homework #4  
 Economics 411  
 Due Wednesday, October 29  
 Christiano

1. Consider the handout on the course website, ‘Recursive Methods and Participation Constraints’. Do questions 1-3 at the end (<http://faculty.econ.northwestern.edu/faculty/christiano/d11/d1105/incentive.pdf>).
2. The  $Ak$  model leads one to consider the following functional equation:

$$T(v) = \max_{0 \leq \lambda \leq A+1-\delta} \frac{[A+1-\delta-\lambda]^{(1-\sigma)}}{1-\sigma} + \beta\lambda^{(1-\sigma)}v.$$

Suppose  $\sigma > 1$ ,  $A > \delta$ , and  $\beta(A+1-\delta)^{1-\sigma} < 1$ .

- (a) Show:  $T(v) = \infty$  for  $v > 0$ ,  $T(0) = \frac{[A+1-\delta]^{(1-\sigma)}}{1-\sigma}$ .
- (b) Show: the derivative of  $T$  at  $v = v_0 < 0$  is:

$$\frac{dT(v_0)}{dv} = \beta\lambda(v_0)^{(1-\sigma)},$$

where

$$\lambda(v_0) = \operatorname{argmax}_{0 \leq \lambda \leq A+1-\delta} \frac{[A+1-\delta-\lambda]^{(1-\sigma)}}{1-\sigma} + \beta\lambda^{(1-\sigma)}v_0.$$

- (c) Show that  $T$  does not satisfy the conditions of Blackwell’s theorem.
- (d) What happens to  $\lambda(v)$  as  $v \rightarrow -\infty$ ?
- (e) What does the graph of  $T(v)$  versus  $v$  for  $v \leq 0$  look like? Does it cross a  $45^\circ$  line drawn in the negative orthant? Draw the  $T(v)$  function by hand, emphasizing its qualitative features.
- (f) Explain, using the graph you just developed, why  $T^j v_0 = v^*$  as  $j \rightarrow \infty$ , for every  $v_0 < 0$ . Also, explain why  $v^*$  such that  $v^* = T(v^*)$ , is unique.

3. The following three sector exogenous growth model was proposed in Kongsamut, Rebelo and Xie (see their paper, ‘Beyond Balanced Growth’, on the course web site), to explain several key features of long-run (i.e., 1869-1990) growth: (i)  $K/Y$  is roughly constant, where  $K$  denotes the aggregate stock of capital, and  $Y$  denotes aggregate output, (ii)  $K$  grows at a roughly constant rate, (iii) the rate of return on capital is relatively constant, and (iv) resources have been reallocated out of agriculture and into services, while manufacturing has a relatively stable (truth is, manufacturing is falling a little) share in the economy. Consider the following technology. Agricultural output,  $A_t$ , is produced using the following production function:

$$A_t = B_A K_{At}^\alpha (N_{At} z_t)^{1-\alpha},$$

where  $B_A > 0$ ,  $0 < \alpha < 1$ ,  $z_t$  denotes the state of technology, and  $K_A$  and  $N_A$  denote capital and labor allocated to agriculture. The state of technology evolves according:

$$z_t = \exp(g) z_{t-1}, \quad g > 0.$$

The manufacturing sector produces output that can be converted into consumption goods,  $C_t$ , or capital goods:

$$C_t + K_{t+1} - (1 - \delta)K_t = K_{Mt}^\alpha (N_{Mt} z_t)^{1-\alpha},$$

in obvious notation. Finally, services,  $S_t$ , are produced according to:

$$S_t = B_S K_{St}^\alpha (N_{St} z_t)^{1-\alpha}.$$

Suppose that at a point in time, households supply  $K_t$  units of capital to the capital rental market and 1 unit of labor to the labor markets, so that clearing in these markets requires:

$$N_{At} + N_{Bt} + N_{Mt} = 1, \quad K_{At} + K_{Bt} + K_{Mt} = K_t.$$

Prices are denominated in units of manufactured goods, so that the price of a manufactured good is unity. Let the price, in units of manufactured goods, of an agricultural good, be  $P_{At}$ . Let the price of a service be  $P_{St}$ . Finally, let  $r_t$  and  $w_t$  denote the rental rate and wage rate. Suppose that the three technologies are operated by competitive firms.

- (a) Show that competitive behavior by firms implies:  $K_{At}/N_{At} = K_{Mt}/N_{Mt} = K_{St}/N_{St} = K_t$ ,  $P_{At} = 1/B_A$ ,  $P_{St} = 1/B_S$ ,

$$C_t + K_{t+1} - (1 - \delta)K_t + \frac{A_t}{B_A} + \frac{S_t}{B_S} = K_t^\alpha z_t^{1-\alpha}. \quad (1)$$

One can treat the latter as a ‘reduced form’ expression for the economy’s resource constraint.

- (b) Derive an expression for the rate of return on capital. If  $K_t$  grows at the rate,  $g$ , will the rate of return on capital be constant?
- (c) Suppose preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \frac{[(A_t - \bar{A})^\eta C_t^\gamma (S_t + \bar{S})^\theta]^{1-\sigma}}{1 - \sigma}, \quad \eta + \gamma + \theta = 1, \quad \eta, \gamma, \theta, \sigma > 0.$$

- i. Write out a budget constraint for households and define a sequence of markets equilibrium for this economy.
- ii. Derive the household’s intertemporal Euler equation associated with capital.
- iii. Show that household optimization, together with the results for prices you derived above, imply:

$$\frac{\gamma(A_t - \bar{A})}{\beta C_t} = B_A, \quad \frac{\gamma(S_t + \bar{S})}{\theta C_t} = B_S. \quad (2)$$

- iv. Substitute out for  $A_t - \bar{A}$  and  $S_t + \bar{S}$  in terms of  $C_t$  in the household’s intertemporal Euler equation, to get an expression in terms of the growth rate of  $C_t$  and the rate of return on capital alone.
- v. Show that in general, there is no reason to expect the economy to converge to a ‘balanced growth path’, i.e., one in which  $A_t$ ,  $C$ ,  $S_t$ ,  $K_t$ ,  $Y_t$  all grow at a constant rate. (Here,  $Y_t = K_t^\alpha z_t^{1-\alpha}$ .) Hint: note that if you scale  $A_t$ ,  $M_t$ ,  $K_t$ ,  $S_t$  by  $z_t$  and work with the scaled versions of (2) and (1) and the household’s intertemporal Euler equation, you cannot get rid of  $z_t$ .

vi. Consider the following restriction:

$$\bar{A}B_S = \bar{S}B_A. \quad (3)$$

Show that in this case, the economy boils down to the one sector growth model with disembodied technical change. Hint: note that in this case, you can replace  $A_t$  and  $S_t$  in (1) by  $A_t - \bar{A}$  and  $S_t - \bar{S}$ , respectively without changing (1). Then, define  $a_t = A_t - \bar{A}$  and  $s_t = S_t - \bar{S}$  everywhere and impose (2), that  $a_t$  and  $s_t$  are each proportional to  $M_t$ .

vii. Explain why it is that under (3), the following are true:

$$\frac{K_t}{z_t} \rightarrow k^*, \quad \frac{A_t - \bar{A}}{z_t} \rightarrow a^*, \quad \frac{S_t + \bar{S}}{z_t} \rightarrow s^*$$

$$\frac{K_{t+1}}{K_t}, \frac{Y_{t+1}}{Y_t} \rightarrow \exp(g),$$

where  $k^*$ ,  $a^*$ ,  $s^*$  are finite, positive, constants.

viii. Provide a simple formula for  $k^*$ .

ix. What happens to the rate of return on capital along a growth path? What happens to the distribution of employment between sectors along a growth path?

x. What does the model imply for  $P_{k^t}$ , the consumption price of capital, along the growth path.