Homework #5
Economics 411
Due Wednesday, November 5
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1. (Vintage capital interpretation of exogenous, embodied technical change model.) Consider an economy with capital of different vintages. At time t, the amount of capital of vintage τ, $k_{t,\tau}$, $\tau = 1, 2, 3, ...$, is

$$k_{t,\tau} = \gamma^{t-\tau}(1-\delta)^{\tau-1} i_{t-\tau},$$

where $\gamma > 1$, $0 < \delta < 1$, $i_{t-\tau}$ is the amount of investment, in time $t-\tau$ consumption units, applied in period $t-\tau$. Capital which has vintage τ in period t has vintage $\tau+1$ in period $t+1$. Investment expenditures at time t, i_t, must all be applied to the latest vintage (for a model in which investment in old vintages is feasible and desirable, see Chari and Hopenhayn, JPE, 1991) and results in $k_{t+1,1} = \gamma^t i_t$ units of new-vintage period $t+1$ installed capital goods. Consider a given amount of investment, i. Note that this investment applied in period $t+1$ produces more new-vintage installed capital (i.e., $\gamma^{t+1} i$) than the same level of investment applied in period t (i.e., $\gamma^t i$). This reflects the assumption, $\gamma > 1$ which is designed to capture the notion that there is exogenous technical progress that is embodied in new capital equipment. Note that the efficiency of a particular vintage stays constant over time, it's just that the efficiency of each succeeding vintage is greater than the efficiency of the previous one.

Capital of each vintage is operated with labor to produce a homogeneous output good, $y_{t,\tau}$ according to the following production function:

$$y_{t,\tau} = k_{t,\tau}^{\alpha} n_{t,\tau}^{1-\alpha}, \ 0 < \alpha < 1, \ \tau = 1, 2, 3, ... \ .$$

Suppose there is a competitive market in capital of different vintages and in labor. Each vintage of capital has the same rental rate, r_t, since capital is measured in common efficiency units. Similarly, the wage rate is w_t.

(a) Show that a firm's profit maximizing choice of $n_{t,\tau}$ gives rise to the following relationships:

$$y_t = k_t^{\alpha} n_t^{1-\alpha},\ (1-\alpha)\left(\frac{k_t}{n_t}\right)^{\alpha} = w_t,\ \alpha\left(\frac{k_t}{n_t}\right)^{\alpha-1} = r_t,$$

where

$$y_t = \sum_{\tau=1}^{\infty} y_{t,\tau},\ k_t = \sum_{\tau=1}^{\infty} k_{t,\tau},\ n_t = \sum_{\tau=1}^{\infty} n_{t,\tau}.$$

(b) Show that 'aggregate capital', k_t, evolves as in the Sollow II model:

$$k_{t+1} = (1-\delta)k_t + \gamma^t i_t.$$

2. Consider the endogenous growth model with human capital discussed in class. One sector produces a homogeneous output good, which is transformed one-for-one into consumption and investment. The homogeneous output good is itself produced using a Cobb-Douglas production function:

$$c_t + k_{t+1} - (1-\delta)\,k_t = k_t^{\alpha} n_t^{1-\alpha}.$$

Another sector produces human capital according to the following accumulation equation:

$$h_{t+1} = h_t + \lambda\,(h_t - n_t)\,,$$

where $\lambda > 0$, $c_t \geq 0$, $k_{t+1} \geq (1-\delta)\,k_t$, $0 \leq n_t \leq h_t$, and h_0, k_0 are given. Preferences are:

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma},$$

$\gamma > 0$. To ensure boundedness, we require $\beta\,(1+\lambda)^{1-\gamma} < 1$. In class, the problem was reformulated in recursive form. It was shown that there are policy rules of the form, $x_{t+1} = f\,(x_t)\,,\ y_t = g\,(x_t)\,,$ where $x_t = k_t/h_t$ and $y_t = h_{t+1}/h_t$. Set $\alpha = 1/3$, $\delta = 0.10$, $\beta = 0.97$, $\lambda = 0.04$, $\gamma = 1.1$. Compute steady state values of x, y. How do these values change with α and with λ? Provide intuition.

3. It may seem puzzling that in the OLG model presented in class, there is no growth in capital, regardless of how high the rate of return on capital is. The resolution to the puzzle is that each agent can save a lot over its lifetime, even if capital is not growing in the aggregate. Thus, suppose b is large so that young people have a strong incentive to save. Display a set of equations that allow you to compute a steady state equilibrium (i.e., one in which all old consume the same amount, all young compute the same amount, the capital stock is constant and so are the rental rate and wage rate) with positive growth in the consumption of each household.

4. Consider the model economy associated with Romer's model of growth through specialization. That is, preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}, \gamma > 0.$$

The technology for producing final goods is:

$$y_t = \int_0^{M_t} x_t(i)^\alpha di, \ M_t > 0, \ 0 < \alpha < 1. \tag{1}$$

To produce $x_t(i)$ units of the i^{th} intermediate good requires

$$\frac{1}{2}(1 + x_t(i)^2) \tag{2}$$

units of capital if $x_t(i) > 0$ and zero units of capital if $x_t(i) = 0$. The following constraint must be satisfied:

$$\int_0^{M_t} \frac{1}{2}(1 + x_t(i)^2) di = k_t, \tag{3}$$

where k_t is the beginning-of-period t aggregate stock of capital. The initial capital stock, $k_0 > 0$, is given. The resource constraint is:

$$c_t + I_t \leq y_t, \tag{4}$$

and the aggregate capital accumulation technology is given by:

$$k_{t+1} = (1 - \delta)k_t + I_t.$$

The efficient allocations for this economy solve the planning problem, maximize utility with respect to $\{M_t, k_{t+1}, y_t, c_t, x_t(i), i \in (0, M_t)\}_{t=0}^{\infty}$, subject to the various constraints.

(a) Explain why economic efficiency dictates $x_t(i) = \bar{x}_t$ for $i \in (0, M_t)$. From here on, you may simply assume $x_t(i) = \bar{x}_t$ for all $i \in (0, M_t)$.

(b) Show that the planning problem for the Romer economy coincides with the planning problem for the Ak model. In particular, show that the problem can be written,

$$max_{\{k_{t+1} \in \Gamma(k_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1}),$$

where

$$F(k, k') = \frac{\left[(A + 1 - \delta)\, k - k'\right]^{1-\gamma}}{1 - \gamma} = \max_{\bar{x}_t, M_t} \frac{c_t^{1-\gamma}}{1 - \gamma}.$$

The last maximization is subject to (1)-(4), and the given values of k_t, k_{t+1}. Display an expression for the value of A in terms of model parameters. In addition to verifying the form of F, show what the constraint set, Γ, is.

(c) Identify a set of parameter values under which positive growth is efficient, although the growth rate in the market decentralization analyzed in class is zero.

(d) The problem with monopoly power is that it results in an inefficiently low level of activity (in the Romer model, the root of this inefficiency is the monopoly power that leads monopolists to pay a rental rate on capital that is less than its social marginal product). In the Romer model we have just seen that this manifests itself in the form of inefficiently low growth. The pace at which new varieties of specialized inputs (e.g., specialized manufactured goods, specialized labor) are introduced is too slow in the market economy. Some sort of intervention in the market economy is desirable. One possibility is to subsidize the activities of monopolists. Accordingly, let $p(i)x(i)$ be the revenues of the i^{th} monopolist in the absence of taxes or subsidies. A subsidy rate, τ_t, raises the revenues of the i^{th} monopolists to $p(i)x(i)(1 + \tau_t)$. The total cost, G_t, to the government of this subsidy scheme is

$$G_t = \int_0^{M_t} p(i)x(i)\tau_t di.$$

Suppose G_t is financed by a lump sum tax applied to households. That is, the household budget constraint is modified as follows:

$$c_t + k_{t+1} - (1-\delta)k_t = r_t k_t + w_t n_t - T_t,$$

where T_t represents taxes paid by the representative household to the government. Suppose the government balances its budget period by period:

$$T_t = G_t.$$

Find the subsidy rate, τ_t, that causes the allocations in the market economy to coincide with the efficient allocations.

These results have to be interpreted with caution. You have identified an ideal form of government intervention, which makes the private market economy efficient. However, the intervention we investigated abstracts from any social inefficiencies that may be induced by the fact that the subsidy to monopolists have to be financed with taxes. We abstracted from this by assuming that the tax on households is administered in lump-sum form. In practice, such taxes are not available. The only taxes we have in actual economies are attached to specific economic activities (like the income tax) and so they distort those specific activities. So, the problem of 'fixing' the inefficiency in the Romer model is actually more complicated than this question makes it out to be.