Homework #6
Economics 411, Fall 2008
Due Tuesday, November 25.
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1. Consider a model in which a final good is produced using intermediate goods. The final good, y, is produced by a competitive, representative firm using the following homogeneous technology:

$$y = \exp \int_0^1 [\log y_j]\, dj.$$

The firm maximizes profits:

$$y - \int_0^1 p_j y_j dj,$$

taking p_j as given. Here, the price of the final good has been normalized at unity. The j^{th} intermediate good is produced by a monopolist using the following technology:

$$\begin{aligned} y_j &= \begin{cases} f(k_j, l_j) - \phi & f(k_j, l_j) \geq \phi \\ 0 & \text{otherwise} \end{cases}, \qquad (1) \\ f(k_j, l_j) &= k_j^{\alpha} l_j^{1-\alpha},\ 0 < \alpha < 1. \end{aligned}$$

Thus, if the monopolist is to sell y_j units of goods, they must produce the fixed quantity, ϕ, first. The monopolist is competitive in the market for labor and capital and takes the rental rate on capital, r, and the wage rate, w, as given.

(a) Derive the demand curve for the j^{th} intermediate good. Consider the profit maximization problem of the j^{th} intermediate good firm. Show that it has no solution. That is, for any finite price-quantity pair on the demand curve, profits are always increased by increasing the price level.

(b) Suppose that there are other potential entrants into the production of the j^{th} intermediate good, and that they have access to the same technology, (1). Explain why this implies that the profits of the intermediate good producer must be zero.

(c) Show that cost minimization by the j^{th} intermediate good producer, linear homogeneity of f, and the zero profit condition imply that output can be written

$$y_j = \frac{1}{\mu_j} f(k_j, l_j),$$

where μ_j is the firm markup, the ratio of price to marginal cost, λ_j:

$$\mu_j = \frac{p_j}{\lambda_j}.$$

(d) Show that the zero profit condition implies the markup must fall when the firm produces more output. Provide the intuition for this result.

(e) Explain why it is that in equilibrium, final output has the following representation:

$$y = \frac{1}{\mu} f(k, l),$$

where l is household labor supply, k is the supply of capital by households, and μ is the markup. Suppose that k, l and y fluctuate in response to shocks outside of the firm sector. Explain how. according to this theory, conventional empirical measures of disembodied technical change are severely misled. This type of theory is sometimes referred to as a 'theory of TFP' (Total Factor Productivity, another name for disembodied technology). It's a theory about how conventional measures of TFP may be recovering some kind of endogenous variable, rather than true, exogenous technology.

2. Consider an economy in which final output is produced by a perfectly competitive firm, which uses intermediate inputs, Y_{it}, $i \in (0, 1)$:

$$Y_t = \left[\int_0^1 Y_{it}^{\rho} di \right]^{\frac{1}{\rho}}, \; 0 < \rho \leq 1.$$

The price of the i^{th} input is p_{it}, and the output price is p_t. The firm's problem is to maximize profits:

$$p_t Y_t - \int_0^1 p_{it} Y_{it} di,$$

taking all prices parametrically. This leads to the following first order condition:

$$Y_{it} = Y_t \left(\frac{p_t}{p_{it}} \right)^{\frac{1}{1-\rho}}, \; i \in (0,1).$$

Substituting this back into the final goods production function:

$$p_t = \left[\int_0^1 p_{it}^{\frac{\rho}{\rho-1}} di \right]^{\frac{\rho-1}{\rho}}$$

Each intermediate good is produced by a single producer, who sets price equal to marginal cost because of the existence of a competitive fringe. Any intermediate good firm that attempted to set a higher price would be bumped out of the market. Each intermediate good firm has a linear production function in labor, with marginal productivity equal to unity. What differentiates the intermediate good firms is that those with $i \in (0, \alpha)$ must borrow the wage bill in advance at gross rate of interest, R_t, while the rest can finance the wage bill out of receipts. Those firms have no financing requirements. As a result, the marginal cost of a unit of labor for firms, $i \in (0, \alpha)$ is $w_t R_t$ and the marginal cost of a unit of labor is w_t for the rest.

You should take the aggregate supply of labor by households, L, as a given number.

(a) Derive an expression for the output of final goods that has the following form:

$$Y = \phi(R) L,$$

provide a simple, closed form expression for $\phi(R)$. Show that $\phi(1) = 1$, $\phi'(1) = 0$. Evidently, the heterogeneous borrowing requirements of different agents has the potential to supply a theory of TFP.

(b) Consider a jump in the interest rate from $R = 1.05$ to 1.10. Is there a value of α or ρ that will associate this jump in R with something like a 10 percent drop in efficiency?

3. Following is a deterministic economy composed of one representative, competitive household, and a representative, competitive firm. Preferences of the household are given by:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t), \;\; u(c, n) = \log c + \sigma \log(1 - n),$$

where c_t denotes consumption and n_t denotes hours worked. The household budget equation is

$$c_t + I_t \leq w_t n_t + r_t k_t,$$

where w_t is the wage rate and r_t is the rental rate on capital, k_t. Here, I_t denotes investment, which the household applies to increasing the stock of capital, using the following technology:

$$k_{t+1} = (1 - \delta)k_t + I_t, \;\; 0 < \delta < 1.$$

The household must satisfy $c_t, k_{t+1} \geq 0$ and $0 \leq n_t \leq 1$, for all t. In addition, the household's initial stock of capital, k_0, is given.

The representative firm has access to the following technology:

$$y = Y^{\gamma} k^{\alpha} n^{1-\alpha}, \;\; \gamma = 1 - \alpha, \;\; \alpha = 1/3,$$

where Y is economy-wide average output, and y, k, n are firm output, capital, and employment, respectively. Note that the firm has constant returns to scale in the variables that it controls directly. Note too, the 'externality' in this production function. If all other firms are producing a lot (i.e., Y is big), this raises the productivity of an individual firm. Firms maximize profits given r_t and w_t. In equilibrium, $Y = y$.

(a) Define a sequence-of-markets equilibrium for this economy.

(b) Establish the necessity of the Euler equations for labor and capital:

$$w_t = -\frac{u_{n,t}}{u_{c,t}}, \;\; u_{c,t} = \beta u_{c,t+1} \left[r_{t+1} + 1 - \delta \right],$$

where $u_{x,t}$ is the partial derivative of u with respect to $x = c_t, n_t$. (Hint: set up the household problem in Lagrangian form, derive the first order necessary conditions for optimality and substitute out the multipliers.)

(c) Establish the following result. Suppose there is are sequences, $\{n_t\}$, $\{c_t\}, \{k_{t+1}\}$, which satisfy the inequality constraints, the household's budget constraint, the Euler equations and the transversality condition:

$$\lim_{T\to\infty} \beta^T u_{c,T}\left[r_{t+T} + 1 - \delta\right] k_T = 0.$$

Show that these sequences generate higher utility for the household than any other sequences which satisfy the household's budget constraint at each date. (Hint: follow the same proof strategy as for Theorem 4.15. In particular, write out D, the difference between the candidate optimal allocations and an arbitrary alternative feasible allocation. Apply concavity of the utility function, and then substitute out all but the first term in D using the budget constraint and first order conditions. Finally, use the transversality condition in the limit as $T \to \infty$ to establish $D \geq 0$.)

(d) Verify that, in an equilibrium where firms are all identical, so that $y = Y$:

$$r_t = \alpha n_t^2,\ w_t = \gamma k_t n_t,\ c_t + k_{t+1} - (1-\delta)k_t = k_t n_t^2.$$

(e) Show that by combining the household and firm Euler equations, one obtains:

$$\frac{n_t^2 + 1 - \delta - \frac{\gamma}{\sigma} n_t(1-n_t)}{n_t(1-n_t)} = \beta \frac{\alpha n_{t+1}^2 + 1 - \delta}{n_{t+1}(1-n_{t+1})},\ t = 0, 1, 2, ... \quad (2)$$

Let (2) be represented as $v(n_t, n_{t+1}) = 0$. Note that this implicitly defines a map from n_t to n_{t+1}. Show that this map is composed of two functions, $n_{t+1} = f_i(n_t)$, $i = 1, 2$, where $f_1 > f_2$ for all n_t. Display analytic expressions for these functions. (Hint: remember the formula for the roots of a second order polynomial.)

(f) Prove the following result. 'Suppose a sequence, $n_0, n_1, ...$ is found which satisfies (2) and also has the property, $a \leq n_t \leq b$ for $a > 0$ and $b < 1$ for all t. Then that sequence corresponds to an equilibrium.'

(g) Set $\delta = 0.02$ and identify a value of σ so that the higher of the two values of n which set $v(n, n) = 0$ has the property, $n = 1/3$. What is the lower value of n such that $v(n, n) = 0$? Show that the stationary equilibrium, $n_0 = n_1 = ... = 1/3$ is indeterminate. Construct a sunspot equilibrium in a neighborhood of the indeterminate steady state. Display graphs with simulations of employment, capital, the rental rate of capital, the wage rate and consumption.

(h) Compute the efficient level of employment and growth for the economy. Compute the v function corresponding to the equilibrium in the version of the economy which implements the efficient 'automatic stabilizer tax rate'.