

1 Introduction

These notes describe a version of the model presented by Comin and Gertler (2006) which integrates a theory of cycles and growth. The cycles they have in mind are ‘intermediate cycles’, which are frequencies of oscillation in the data that are lower than the 8 to 32 quarter frequencies conventionally associated with business cycles.

In some respects, the model resembles the model of Romer that we studied earlier, in which growth occurs because of the benefits of specialization. In the model discussed here, growth occurs because of the benefits of new ideas for production. The notion is that there are three types of ideas: (i) those which have, with the application of economic resources, been turned into practical use; (ii) those which are known, but which require further development to be made practical and (iii) the ideas that are out there, still waiting to be discovered.¹ Examples of (i) include the internal combustion engine, the electric light bulb and the polio vaccine. Examples of (ii) include a fully functioning artificial human heart, and (iii) include the lap top computer, from the perspective of, say, 50,000 BCE.

The following section describes the model. After that, evidence is reported on the dynamic properties of the model. In the concluding remarks that appear in the third section, we summarize some of the research implications of this handout.

2 Model

We begin with some general remarks describing the overall structure of production, and how it relates to the ideas mentioned in the introduction. After that, we describe the firm and household sectors.

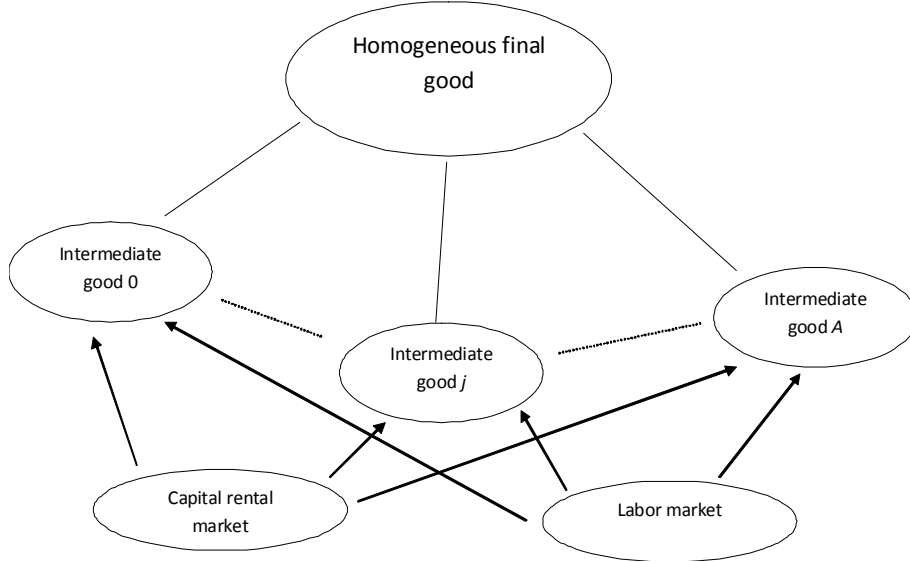
2.1 Ideas, Production and Total Factor Productivity

At date t , the complete set of ideas, those known and as-yet unknown, are identified with the points on the positive real line. The points between 0 and $A_t > 0$ correspond to the ideas that are in practical use, the points between A_t and Z_t correspond to ideas that are known, but which still require development to be turned into something practical. Finally, the points on the interval, (Z_t, ∞) , correspond to the ideas that are not yet known. Each idea, or point

¹Implicitly, we assume that ‘ideas’ are objects that exist independently of the person who may perceive an idea. The notion of an idea that we use is that it is a true characterization of some aspect of the essence of things.

on the positive line, corresponds to a technology for producing an intermediate good. The intermediate good is useful for producing a final homogeneous good, using a Dixit-Stiglitz production technology. The more ideas that are in use, that is, the larger is A_t , the more that can be produced for a given total quantity of aggregate capital and labor resources (see Figure 1). The latter are supplied to the firm sector by households.

Figure 1: The Organization of Production



To see the impact of A_t on productivity, here is the final good production function:

$$Y_t = \left[\int_0^{A_t} Y_t(j)^{\frac{1}{\theta}} dj \right]^{\theta}, \quad \theta > 1, \quad (1)$$

where $Y_t(j)$ denotes the quantity of intermediate goods and Y_t denotes the quantity of final goods. The production function for the j^{th} intermediate good is:

$$Y_t(j) = K_t(j)^{\alpha} L_t(j)^{1-\alpha}, \quad 0 < \alpha < 1, \quad (2)$$

where $K_t(j)$ and $L_t(j)$ denote the quantity of capital and labor, respectively, used in the production of intermediate good $j \in (0, A_t)$. The link between capital and labor used at the intermediate good level and the amount available at the aggregate level, K_t and L_t , is

$$K_t = \int_0^{A_t} K_t(j) dj = A_t \tilde{K}_t, \quad L_t = \int_0^{A_t} L_t(j) dj = A_t \tilde{L}_t,$$

where $K_t(j) = \tilde{K}_t$ and $L_t(j) = \tilde{L}_t$ for $j \in (0, A_t)$ denote levels of capital and labor used at the plant level. Substituting into the aggregate production function:

$$\begin{aligned} Y_t &= A_t^{\theta} \tilde{K}_t^{\alpha} \tilde{L}_t^{1-\alpha} = A_t^{\theta} \left(\frac{K_t}{A_t} \right)^{\alpha} \left(\frac{L_t}{A_t} \right)^{1-\alpha} \\ &= A_t^{\theta-1} K_t^{\alpha} L_t^{1-\alpha}, \end{aligned}$$

so that

$$Y_t = K_t^\alpha (z_t L_t)^{1-\alpha}, \quad (3)$$

where

$$z_t = A_t^{\frac{\theta-1}{1-\alpha}}. \quad (4)$$

As we shall see, the equilibrium growth rate of the economy is the growth rate of z_t :

$$\mu_{z,t} \equiv \frac{z_t}{z_{t-1}}.$$

Note from (3) how the increase in the range of practical ideas in the economy has the same impact on the production function as an increase in exogenous technology does in the real business cycle model. An important difference is that here z_t is an endogenous variable.² In this respect, this model is a response to Prescott's (1998) challenge, 'Needed: a theory of total factor productivity (TFP)'. In virtually every substantial movement in observed output, measured TFP moves in the same direction with output. For example, in the Asian currency crisis of the late 1990s, the collapse in output was associated with a collapse in TFP. Also, during business cycle recessions, measured TFP falls and then it rises in business cycle expansions. In each of these cases, very few researchers take seriously the idea that these movements in TFP are literally exogenous shocks as posited in the early real business cycle models. Thus, most researchers take seriously Prescott's challenge to construct a model which has the implication that TFP comoves with output and in which the movements in TFP are endogenous. In our model, measured TFP is endogenous, because it reflects the decisions of firms to develop new ideas. Presumably, the model is at best only a small step in the direction of a fully successful response to Prescott's challenge. For example, it may be implausible to suppose that the fall in TFP in the Asian currency crisis reflects the sort of considerations encompassed by A_t .

2.2 Law of Motion of Ideas

We derive the law of motion for ideas in terms of scaled variables. The frontier of goods, Z_t , evolves according to the following law of motion:

$$Z_{t+1} = \exp(\chi_t) Z_t \quad (5)$$

where

$$\chi_t = (1 - \rho)\bar{\chi} + \rho\chi_{t-1} + u_t, \quad |\rho| < 1, \quad (6)$$

²The standard measurement of total factor productivity (TFP) uses a measure of aggregate GDP, Y_t , a measure of the aggregate stock of capital, K_t and of the aggregate quantity of labor, L_t , and computes TFP as follows:

$$TFP = \frac{Y_t}{K_t^\alpha L_t^{1-\alpha}},$$

where α is estimated from data on the share of income going to capital. In the model used here, TFP is $z_t^{1-\alpha}$.

and u_t is a mean-zero, i.i.d. shock.

The points in the interval, $Z_t - A_t$, represents the set of potential new technologies. One can attempt to develop an atom in that interval into a practical technology through the expenditure of a quantity of final goods, h_t .³ In this way a potential technology is developed with probability

$$\lambda_t = \lambda(\Gamma_t h_t), \quad \Gamma_t = \frac{A_t}{o_t}.$$

Here, $\lambda(\cdot)$ is increasing and concave, $\lambda' > 0$ and $\lambda'' < 0$. The presence of A_t in Γ_t captures the notion of ‘learning by doing’. A higher current value of A_t means that to activate a given good with a specific probability requires the application of fewer resources, h_t . The object, o_t is present in this expression in order to help ensure $\Gamma_t h_t$ converges to a constant in steady state. We set o_t as follows:

$$o_t = bK_t,$$

where $b > 0$ is a parameter that is chosen to ensure a desired value of λ in steady state. Let,

$$o_{z,t} \equiv \frac{o_t}{z_t} = \frac{bk_t}{\mu_{z,t}}, \quad h_{A,t} \equiv \frac{h_t A_t}{z_t}.$$

The objects, $o_{z,t}$, and $h_{A,t}$ converge in nonstochastic steady state. Then, in terms of scaled variables:

$$\lambda_t = \lambda\left(\frac{h_{A,t}}{o_{z,t}}\right).$$

The law of motion of A_t is as follows:

$$A_{t+1} = \lambda_t [Z_t - A_t] + A_t.$$

That is, the set of developed ideas in period $t + 1$ is equal to the set in period t plus the fraction of the ideas in $Z_t - A_t$ that have been successfully developed into practical ideas during period t . Let

$$a_t \equiv \frac{A_t}{Z_t}.$$

Dividing the law of motion of A_t by Z_{t+1} :

$$\frac{A_{t+1}}{Z_{t+1}} = \frac{\lambda_t [Z_t - A_t]}{Z_{t+1}} + \frac{A_t}{Z_{t+1}}$$

or,

$$a_{t+1} = \frac{\lambda(h_{A,t}/o_{z,t})}{\exp(\chi_t)} + \frac{1 - \lambda(h_{A,t}/o_{z,t})}{\exp(\chi_t)} a_t. \quad (7)$$

In sum, the value of A_t is increased by the appropriate application of resources for development. The object, Z_t , increases over time according to an exogenous stochastic process. In principle, one could make Z_t endogenous too, by supposing that it can be made to increase more rapidly by devoting resources to basic research (as in Comin and Gertler).

³Actually, the amount of resources applied to an individual atom is the infinitesimal quantity, $h_t dj$.

2.3 Equilibrium Conditions Associated with the Firm Sector

Competitive, final good producers in period t take A_t as given and produce output using (1). The profit maximizing choice of $Y_t(j)$ leads to the following first order condition:

$$P_t(j) = P_t \left(\frac{Y_t(j)}{Y_t} \right)^{\frac{1-\theta}{\theta}}, \quad (8)$$

for $j \in (0, A_t)$. Expression (8) is treated as the demand curve by the producer of the j^{th} intermediate good, $j \in (0, A_t)$. Combining (1) and (8) implies:

$$P_t = \left[\int_0^{A_t} P_t(j)^{\frac{1}{1-\theta}} \right]^{1-\theta}. \quad (9)$$

We now consider the problem of the manager of intermediate goods, $j \in (0, A_t)$ and $j \in (A_t, Z_t)$.

2.3.1 Production Decision of Active Intermediate Goods Producers

Intermediate good producer $j \in (0, A_t)$ uses the Cobb-Douglas production function in (2). As noted earlier, the j^{th} plant is a monopolist in the production of the j^{th} intermediate good and it is competitive in factor markets. Let r_t and \tilde{w}_t denote the rental rate of capital and the wage rate, respectively, denoted in units of the final output good. Let the marginal cost of production, in units of the final good, be denoted by \tilde{s}_t . A well known result for Cobb-Douglas production functions is

$$\tilde{s}_t = \left(\frac{r_t}{\alpha} \right)^\alpha \left(\frac{\tilde{w}_t}{1-\alpha} \right)^{1-\alpha},$$

denoted in units of the final output good. The monopolist solves:

$$\max_{P_t(j), Y_t(j)} \frac{P_t(j)}{P_t} Y_t(j) - \tilde{s}_t Y_t(j),$$

subject to the demand curve, (8). As is known for this type of environment, optimality leads the monopolist to set its price as a fixed markup, θ , over marginal cost. Because marginal cost is denominated in units of the final good, the relevant condition pertains to $P_t(j)/P_t$:

$$\frac{P_t(j)}{P_t} = \theta \tilde{s}_t. \quad (10)$$

Using this expression to substitute out for $P_t(j)$ in (9), we obtain:

$$1 = \theta \tilde{s}_t A_t^{1-\theta} = \theta \tilde{s}_t z_t^{1-\alpha}, \quad (11)$$

using (4). We adopt the following scaling:

$$s_t \equiv \tilde{s}_t z_t^{1-\alpha}, \quad w_t \equiv \frac{\tilde{w}_t}{z_t}.$$

For ease of reference, we report the full set of scaled variables in the appendix. Note from (11) that $1 = \theta s_t$, or,

$$s_t = \frac{1}{\theta}, \quad s_t = \frac{\left(\frac{r_t}{\alpha}\right)^\alpha \left(\frac{w_t}{1-\alpha}\right)^{1-\alpha}}{\epsilon_t}. \quad (12)$$

The firm's cost minimization problem is useful for determining its first order conditions for $K_t(j)$ and $L_t(j)$:

$$\min_{K_t(j), L_t(j)} r_t K_t(j) + \tilde{w}_t L_t(j) + \lambda [Y_t(j) - \epsilon_t K_t(j)^\alpha L_t(j)^{1-\alpha}].$$

The first order conditions for this problem are:

$$\begin{aligned} r_t &= \tilde{s}_t \alpha \epsilon_t \left(\frac{K_t(j)}{L_t(j)} \right)^{\alpha-1} \\ \tilde{w}_t &= \tilde{s}_t (1 - \alpha) \left(\frac{K_t(j)}{L_t(j)} \right)^\alpha, \end{aligned}$$

where we have used the fact that the multiplier, λ , associated with the firm's cost minimization problem is its marginal cost. and expressing things in terms of scaled variables, we obtain

$$\begin{aligned} r_t &= s_t \alpha \left(\frac{k_t(j)}{L_t(j) \mu_{z,t}} \right)^{\alpha-1} \\ w_t &= s_t (1 - \alpha) \left(\frac{k_t(j)}{L_t(j) \mu_{z,t}} \right)^\alpha \end{aligned}$$

where

$$\mu_{z,t} \equiv \frac{z_t}{z_{t-1}} = \left(\frac{a_t \exp(\chi_{t-1})}{a_{t-1}} \right)^{\frac{\theta-1}{1-\alpha}}, \quad k_t(j) = \frac{K_t(j)}{z_t}.$$

Making use of (12) and the fact that all intermediate good firms, $j \in (0, A_t)$, make the same choices:

$$r_t = \frac{\alpha}{\theta} \left(\frac{k_t}{L_t} \frac{1}{\mu_{z,t}} \right)^{\alpha-1} \quad (13)$$

$$w_t = \frac{1 - \alpha}{\theta} \left(\frac{k_t}{L_t} \frac{1}{\mu_{z,t}} \right)^\alpha. \quad (14)$$

We now develop an expression for the discounted profits of the $j \in (0, A_t)$ plants. Period t profits of the j^{th} plant are, in units of the final good:

$$\Pi_t(j) \equiv \frac{P_t(j)}{P_t} Y_t(j) - \tilde{s}_t Y_t(j) = (\theta - 1) \tilde{s}_t Y_t(j),$$

using (10). Using (8), (11) and (12) to substitute out for $Y_t(j)$:

$$\begin{aligned}
\Pi_t(j) &= \frac{P_t(j)}{P_t} Y_t(j) - \tilde{s}_t Y_t(j) = (\theta - 1) \tilde{s}_t Y_t(j) \\
&= (\theta - 1) \tilde{s}_t \left(\frac{P_t(j)}{P_t} \right)^{\frac{\theta}{1-\theta}} Y_t \\
&= (\theta - 1) \tilde{s}_t A_t^{-\theta} Y_t \\
&= \left(\frac{\theta - 1}{\theta} \right) A_t^{-1} Y_t,
\end{aligned}$$

so that

$$\Pi_{A,t} \equiv \Pi_t(j) \frac{A_t}{z_t} = \frac{\theta - 1}{\theta} y_t,$$

where

$$y_t \equiv \frac{Y_t}{z_t} = \frac{K_t^\alpha (z_t L_t)^{1-\alpha}}{z_t} = \left(\frac{k_t}{\mu_{z,t}} \right)^\alpha L_t^{1-\alpha},$$

by (3). Later, we show that y_t converges to a constant in a nonstochastic steady state, so that scaled profits, $\Pi_{A,t}$, do too. As a result, along a steady state growth path, profits are proportional to

$$\frac{z_t}{A_t} = \frac{z_t}{z_t^{\frac{1-\alpha}{\theta-1}}} = z_t^{\frac{\theta+\alpha-2}{\theta-1}}.$$

It follows that the growth rate of profits in steady state is

$$\frac{\theta + \alpha - 2}{\theta - 1} \mu_z, \tag{15}$$

where μ_z denotes the steady state value of $\mu_{z,t}$. If we set α to the reasonable value of $1/3$, then θ would have to exceed $2 - \alpha = 1.67$ for this growth rate to be positive. Such a high value of θ is not reasonable, and so we conclude that for reasonable parameter values, the model has the property that the profits of each individual active plant falls over time. Note that all production processes continue to be used, even with the arrival of new ideas. Thus, this is not an example of a model with Schumpeterian ‘creative destruction’.

The present discounted value of profits for the j^{th} firm is:

$$V_t = \Pi_t(j) + \beta E_t \frac{u_{c,t+1}}{u_{c,t}} \Pi_{t+1}(j) + \beta^2 E_t \frac{u_{c,t+2}}{u_{c,t}} \Pi_{t+2}(j) + \dots$$

Note here that profits at time $t + l$ are discounted by

$$\beta \frac{u_{c,t+l}}{u_{c,t}},$$

an object that is assumed to be treated as an exogenous variable by plants. The discount rate converts a period $t + l$ flow into period t terms using the valuation assigned by households.⁴ This is appropriate given that the household is the owner of the firm which owns the plant.

Rescaling the expression for V_t and rearranging gives us the following recursive relationship for scaled V_t :

$$v_{A,t} = \frac{\theta - 1}{\theta} y_t + \beta E_t \frac{u_{c,t+1}^z}{u_{c,t}^z} \frac{a_t}{a_{t+1} \exp(\chi_t)} v_{A,t+1}. \quad (16)$$

Here,

$$u_{c,t}^z \equiv u_{c,t} z_t, \quad v_{A,t} \equiv V_t \frac{A_t}{z_t}.$$

It can be shown that $v_{A,t}$ converges to a constant in a nonstochastic steady state, so that the present discounted value of the profits of an individual plant fall over time at the rate, (15). In addition, $u_{c,t}^z$ also converges to a constant in a nonstochastic steady state. Below, we assume that $u_{c,t} = 1/C_t$, where C_t denotes household consumption. So, convergence of $u_{c,t}^z$ simply corresponds to convergence of C_t/z_t . The value of all the plants in the interval, $(0, A_t)$, is $A_t V_t$, so that in scaled terms the value of profits among the ideas that have been developed is

$$z_t v_{A,t}.$$

Note that the profits of the active plants all taken together grow at the same rate as Y_t

2.3.2 Development

Consider now the problem of a plant associated with a point in the interval, $(Z_t - A_t)$. Let J_t denote the real value of such a plant. Then,

$$J_t = \max_{h_t} -h_t + \beta E_t \frac{u_{c,t+1}}{u_{c,t}} [\lambda_t V_{t+1} + (1 - \lambda_t) J_{t+1}] \quad (17)$$

That is, the plant applies final output, h_t , to an atom in $(Z_t - A_t)$ and the corresponding idea becomes practical with probability λ_t . With the complementary probability, $1 - \lambda_t$, the development effort is not successful and the idea remains in the development phase. The first order condition associated with the choice of h_t is:

$$1 = \beta E_t \frac{u_{c,t+1}}{u_{c,t}} \lambda' \left(\frac{A_t h_t}{o_t} \right) \frac{A_t}{o_t} (V_{t+1} - J_{t+1})$$

We scale as follows:

$$h_{A,t} \equiv \frac{A_t}{z_t} h_t, \quad o_{zt} \equiv \frac{o_t}{z_t}, \quad v_{A,t} \equiv \frac{A_t}{z_t} V_t, \quad j_{A,t} \equiv \frac{A_t}{z_t} J_t$$

⁴A notationally more intensive way of handling the discounting of future cash flows would specify that firms discount using prices for state-contingent goods. Under the assumptions that households also interact in these state contingent markets, prices would in equilibrium be equal to the intertemporal marginal rate of substitution used in the body of the paper.

Notice how different the steady state behavior of h_t , V_t and J_t are predicted to be than other quantities like aggregate output, Y_t . The latter must be scaled by $1/z_t$ and h_t , V_t and J_t are scaled by multiplying by a bigger quantity, A_t/z_t . The discussion around equation (15) indicates that the latter is actually growing, meaning that h_t , V_t and J_t are on a *declining* growth path: then actually fall in steady state. That this must be so is actually not surprising. We would expect the total amount of resources devoted to development, $(Z_t - A_t)h_t$, the total value of producing firms, $V_t A_t$, and the total value of developing firms, $J_t (Z_t - A_t)$, to be growing at the same rate as Y_t . But, for this to be possible requires, because A_t grows in steady state, that h_t , V_t and J_t grow less rapidly than z_t .

We convert the first order condition for h_t into an expression involving only scaled variables as follows:

$$\begin{aligned}
1 &= \beta E_t \frac{u_{c,t+1}^z}{u_{c,t}^z} \lambda'_t \left(\frac{A_t h_t}{o_t} \right) \frac{z_t A_t}{o_t A_{t+1}} A_{t+1} \left(\frac{V_{t+1} - J_{t+1}}{z_{t+1}} \right) \\
&= \beta E_t \frac{u_{c,t+1}^z}{u_{c,t}^z} \lambda'_t \left(\frac{A_t h_t}{o_t} \right) \frac{z_t A_t}{o_t A_{t+1}} (v_{A,t+1} - j_{A,t+1}) \\
&= \beta E_t \frac{u_{c,t+1}^z}{u_{c,t}^z} \lambda'_t \left(\frac{A_t h_t}{o_t} \right) \frac{z_t a_t Z_t}{o_t a_{t+1} Z_{t+1}} (v_{A,t+1} - j_{A,t+1}) \\
&= \beta E_t \frac{u_{c,t+1}^z}{u_{c,t}^z} \lambda'_t \left(\frac{A_t h_t}{o_t} \right) \frac{a_t}{o_{z,t} a_{t+1} \exp(\chi_t)} (v_{A,t+1} - j_{A,t+1}) \\
&= \beta E_t \frac{u_{c,t+1}^z}{u_{c,t}^z} \lambda'_t \left(\frac{A_t \frac{h_{A,t} z_t}{A_t}}{o_t} \right) \frac{a_t}{o_{z,t} a_{t+1} \exp(\chi_t)} (v_{A,t+1} - j_{A,t+1}) \\
&= \beta E_t \frac{u_{c,t+1}^z}{u_{c,t}^z} \lambda'_t \left(\frac{h_{A,t}}{o_{z,t}} \right) \frac{a_t}{o_{z,t} a_{t+1} \exp(\chi_t)} (v_{A,t+1} - j_{A,t+1})
\end{aligned} \tag{18}$$

Next, we obtain a scaled representation of (17). Let

$$j_{A,t} \equiv J_t \frac{A_t}{z_t}.$$

Then, expressing (17) in terms of scaled variables:

$$\frac{j_{A,t}}{A_t} z_t = -\frac{z_t}{A_t} h_{A,t} + \beta E_t \frac{u_{c,t+1} z_t}{u_{c,t} z_{t+1}} \left[\lambda_t \frac{v_{A,t+1}}{A_{t+1}} z_{t+1} + (1 - \lambda_t) \frac{j_{A,t+1}}{A_{t+1}} z_{t+1} \right].$$

Multiply by A_t/z_t :

$$j_{A,t} = -h_{A,t} + \beta E_t \frac{u_{c,t+1}^z A_t A_{t+1}}{u_{c,t}^z z_{t+1} A_{t+1}} \left[\lambda_t \frac{v_{A,t+1}}{A_{t+1}} z_{t+1} + (1 - \lambda_t) \frac{j_{A,t+1}}{A_{t+1}} z_{t+1} \right],$$

or,

$$j_{A,t} = -h_{A,t} + \beta E_t \frac{u_{c,t+1}^z a_t}{u_{c,t}^z a_{t+1} \exp(\chi_t)} \left[\lambda \left(\frac{h_{A,t}}{o_{z,t}} \right) v_{A,t+1} + \left(1 - \lambda \left(\frac{h_{A,t}}{o_{z,t}} \right) \right) j_{A,t+1} \right],$$

where we take it for granted that the above equation is evaluated at the optimal value of h_t .

The total quantity of final goods used in the development of ideas is:

$$\begin{aligned} & h_t (Z_t - A_t) \\ &= \frac{h_{A,t} z_t}{A_t} (Z_t - A_t) \\ &= z_t \frac{h_{A,t}}{a_t} (1 - a_t). \end{aligned}$$

As expected, this grows at the rate, z_t , because $h_{A,t}$ and a_t converge to constants in steady state. Similarly, the present discounted value of the plants in the interval, $(Z_t - A_t)$, is

$$z_t \frac{j_{A,t}}{a_t} (1 - a_t).$$

Since $j_{A,t}$ and a_t converge to constants in nonstochastic steady state, it follows that the aggregate value of all the firms in the interval, $(Z_t - A_t)$ increases over time at the same rate of the economy.

2.3.3 Value of Not-Yet-Implementable Ideas

Consider now a plant associated with the portion of the real line, (Z_t, ∞) . The new ideas available for development in period $t + 1$ are

$$Z_{t+1} - Z_t.$$

These have value,

$$\begin{aligned} E_t \left[\frac{\beta u_{c,t+1}}{u_{c,t}} J_{t+1} \right] (Z_{t+1} - Z_t) &= E_t \frac{\beta u_{c,t+1}^z z_t}{u_{c,t}^z z_{t+1}} \frac{z_{t+1}}{A_{t+1}} j_{A,t+1} (Z_{t+1} - Z_t) = z_t E_t \frac{\beta u_{c,t+1}^z}{u_{c,t}^z} j_{A,t+1} \left(\frac{Z_{t+1}}{A_{t+1}} - \frac{Z_t}{A_{t+1}} \right) \\ &= z_t E_t \frac{\beta u_{c,t+1}^z}{u_{c,t}^z} \frac{j_{A,t+1}}{a_{t+1}} (1 - \exp(-\chi_t)), \end{aligned}$$

after converting to scaled variables. Note how in this expression, we convert the future flows into period t terms using the valuation of these flows by households. The logic behind this is that the households are the ultimate owners of the plants. Now consider the new goods in period $t + 2$:

$$\begin{aligned} & E_t \frac{\beta^2 u_{c,t+2}}{u_{c,t}} J_{t+2} (Z_{t+2} - Z_{t+1}) \\ &= z_t E_t \frac{\beta^2 u_{c,t+2}^z}{u_{c,t}^z} \frac{j_{A,t+2}}{a_{t+2}} (1 - \exp(-\chi_{t+1})), \end{aligned}$$

and so on. Thus, the aggregate value of all ideas that have not yet arrived is:

$$z_t E_t \sum_{j=1}^{\infty} \frac{\beta^j u_{c,t+j}^z}{u_{c,t}^z} \frac{j_{A,t+j}}{a_{t+j}} (1 - \exp(-\chi_{t+j-1})).$$

It is convenient to express the last expression in recursive form:

$$\begin{aligned}
v_{future,t} &= E_t \left\{ \frac{\beta u_{c,t+1}^z j_{A,t+1}}{u_{c,t}^z a_{t+1}} (1 - \exp(-\chi_t)) \right. \\
&\quad + \frac{\beta^2 u_{c,t+2}^z j_{A,t+2}}{u_{c,t}^z a_{t+2}} (1 - \exp(-\chi_{t+1})) \\
&\quad \left. + \frac{\beta^3 u_{c,t+3}^z j_{A,t+3}}{u_{c,t}^z a_{t+3}} (1 - \exp(-\chi_{t+2})) + \dots \right\} \\
&= E_t \frac{\beta u_{c,t+1}^z j_{A,t+1}}{u_{c,t}^z a_{t+1}} (1 - \exp(-\chi_t)) \\
&\quad + E_t \frac{\beta u_{c,t+1}^z}{u_{c,t}^z} \left[\frac{\beta^1 u_{c,t+2}^z j_{A,t+2}}{u_{c,t+1}^z a_{t+2}} (1 - \exp(-\chi_{t+1})) \right. \\
&\quad \left. + \frac{\beta^2 u_{c,t+3}^z j_{A,t+3}}{u_{c,t+1}^z a_{t+3}} (1 - \exp(-\chi_{t+2})) + \dots \right] \\
&= E_t \frac{\beta u_{c,t+1}^z j_{A,t+1}}{u_{c,t}^z a_{t+1}} (1 - \exp(-\chi_t)) + E_t \frac{\beta u_{c,t+1}^z}{u_{c,t}^z} v_{future,t+1}.
\end{aligned}$$

Here, $v_{future,t}$ is the value of the ideas associated with points greater than Z_t , after multiplication by $1/z_t$. The preceding calculations show:

$$v_{future,t} = E_t \frac{\beta u_{c,t+1}^z}{u_{c,t}^z} \left\{ \frac{j_{A,t+1}}{a_{t+1}} (1 - \exp(-\chi_t)) + v_{future,t+1} \right\}$$

2.3.4 The Stock Market Value of All Firms

The value of all firms is simply the value of the sum of the value of the three types of plants discussed above.

$$z_t \left[v_{A,t} + j_{A,t} \left(\frac{1 - a_t}{a_t} \right) + v_{future,t} \right].$$

Suppose, as recommended by Comin, Gertler and Santacreu (2008), we want to compare this object with an empirical measure of the stock market. For such an identification to make sense, we have to suppose that the stock market represents a suitably weighted sampling of each of our three types of plants. This requires in particular that the firms on the stock market include plants associated with ideas that have not yet been discovered. One interpretation of the ‘firm’ in our model that accomplishes this is as follows. Suppose there are many firms and that each owns a sampling of all the plants on the real line. For simplicity we can assume that all firms are the same, so that the ones that are registered in the stock market represent the same collection of plants as the ones that are not traded. Thus, we think of a firm as an entity that is associated with three types of projects: those that are fully operational now, those that are under development and those that will occur in the

future and which are as-yet unknown. To understand what it means to include the latter in a firm, recall that in practice firms are collections of creative people. Such people will inevitably sprout clever ideas in the future - one cannot even imagine in advance what those ideas will be - which the firm will be able to capitalize on. This fact about the people in a firm is part of what makes a firm valuable. Real-world firms are able to capture the value of newly arriving ideas for their share holders. When an engineer or research scientist invents something while working for a firm, the value of that invention belongs to the firm.

The discussion in the previous paragraph provides the best-case scenario under which it makes sense to compare the ‘stock market’ in the model with the value of the stock market in the data. An alternative scenario is suggested in various papers by Jovanovic and Rousseau which suggest that the stock market may oversample firms that encompass activities that have passed the development phase. The notion is that undeveloped, creative activities arrive in (currently) less well established entities that are not traded in the stock market. Anecdotes that come to mind include Bill Gates working with his friends on Microsoft in their garages. We will investigate the implications of different interpretations of the stock market in our model in the results section below.

2.4 Households

The representative household maximizes the present discounted utility given by:

$$E_t \sum_{i=0}^{\infty} \beta^i \left(\log(C_{t+i}) - \psi \frac{(L_{t+i})^{1+\zeta}}{1+\zeta} \right)$$

where $0 < \beta < 1$, $\zeta > 0$ and $\psi > 0$. The budget constraint is as follows.

$$C_t + K_{t+1} + B_{t+1} = \tilde{w}_t L_t + (1 + r_t - \delta)K_t + R_t B_t + \Pi_t - T_t$$

where B_t is total loans the household makes at $t - 1$ that are payable at t , Π_t is the profits and T_t is lump sum taxes. The first order conditions are:

$$\begin{aligned} \tilde{w}_t &= \psi L_t^\zeta C_t \\ 1 &= \beta E_t \frac{C_t}{C_{t+1}} (1 + r_{t+1} - \delta) \\ R_{t+1} &= \left(\beta E_t \frac{C_t}{C_{t+1}} \right)^{-1}. \end{aligned}$$

Rescaling gives us

$$\begin{aligned} w_t &= \psi L_t^\zeta c_t \\ 1 &= \beta E_t \frac{c_t}{c_{t+1}} \frac{1}{\mu_{z,t+1}} (1 + r_{t+1} - \delta) \\ R_{t+1} &= \left(\beta E_t \frac{c_t}{c_{t+1}} \frac{1}{\mu_{z,t+1}} \right)^{-1}. \end{aligned}$$

Finally, the budget constraint turns out to be the resource constraint:

$$c_t + k_{t+1} - (1 - \delta) \frac{k_t}{\mu_{z,t}} + \frac{h_{A,t}}{a_t} (1 - a_t) + g_t = y_t$$

where $g_t = G_t/z_t$ is the scaled government expenditure. We assume that in steady state, $g = \eta_g y$, where η_g is a parameter.

3 Dynamic Properties of the Model

3.1 Parameters Values and Model Solution

We adopt the following specification of the probability of success when resources, h_t , are applied to development:

$$\lambda(\Gamma_t h_t) = (\Gamma_t h_t)^\sigma, \quad \sigma < 1.$$

The model time period is specified to be quarterly, and the parameters are set as follows:

$$\begin{aligned} \alpha &= 0.36, \quad \theta = 1.3, \quad \beta = \frac{1}{1.03^{-0.25}}, \quad \delta = 0.02, \quad \psi = 1, \quad \zeta = 1, \quad \rho = 0.2, \quad \bar{\chi} = 0.007985 \\ b &= 15000, \quad \sigma = 0.2, \quad \eta_g \equiv \frac{g}{y} = 0.20. \end{aligned}$$

With these parameter values, $a = 0.86$, so that in steady state the share of practical ideas in the set of all known ideas is 86 percent. The output to capital ratio is 9.3, only a little lower than 12, a standard estimate of the US output to capital ratio. The steady state household consumption to output ratio is 0.58, which is quite similar to standard estimates for the US economy. The assumption that government consumption is 20 percent of output matches the average ratio of federal state and local government consumption and investment to GDP.⁵ The steady state ratio of the ‘stock market’ to gdp is very high, 30.4. If plants associated with $j > Z_t$ are excluded from the model’s stock market measure, then the stock market to output ratio in steady state is 16.8. The resources devoted to development are 0.7 percent of gdp. That is,

$$\frac{\frac{h_{A,t}}{a_t} (1 - a_t)}{y_t} = 0.0067.$$

In steady state, the probability of success when the steady state quantity of resources is applied to development is $\lambda = 0.05$, or 5 percent. This means that on average, it takes 20 quarters (5 years) from the time an idea is observed to the time that it becomes practical.

To solve the model, we note that the scaled equilibrium conditions of the model have the form:

$$E \left[v \left(x_t, x_{t+1}, x_{t+2}, \chi_t, \chi_{t+1} \right) \mid \chi_t \right] = \underbrace{0}_{14 \times 1}$$

⁵In principle, it would be better to subtract government investment for this purpose of this calculation.

where v denotes the 14 equilibrium conditions listed in Appendix C and x_t is a 14×1 vector of endogenous variables whose values are determined at time t . This list of variables is also listed in Appendix C. To solve the model, we first replaced v by its linear Taylor series expansion, v^l , about $x_t = x_{t+1} = x_{t+2}$, $\chi_t = \chi_{t+1} = \chi$. We then posited a solution of the following form:

$$x_t = Ax_{t-1} + B\chi_t,$$

where A is a 14×14 matrix and B is 14×1 . We then find A and B that satisfy

$$E \left[v^f(x_t, x_{t+1}, x_{t+2}, \chi_t, \chi_{t+1}) | \chi_t \right] = \underbrace{0}_{14 \times 1}$$

for all possible x_t and χ_t . We verify that A, B are the only solution to this problem in which the eigenvalues of the matrix, A , are less than unity in absolute value. These calculations were all done using the software, Dynare, which can be freely downloaded from the web.

3.2 Dynamic Properties of the Model

We compute the responses of the model to a perturbation in u_t . We start the model in steady state in period $t = -1$ and set $u_0 = 0.10$, $u_t = 0$ for $t > 0$. Given the structure of the χ_t process (see (6)), χ_0 does not change, χ_1 jumps by 0.10, χ_2 jumps by 0.2×0.10 , χ_3 jumps by $0.2^2 \times 0.10$, and so on. According to (5), $\log Z_0$ does not change, but $\log Z_1$ jumps by 0.10, $\log Z_2$ jumps by $0.10 + 0.2 \times 0.10$, $\log Z_3$ jumps by $0.10 + 0.20 \times 0.10 + 0.20^2 \times 0.10$, and so on. Thus, as $t \rightarrow \infty$, $\log Z_t$ eventually jumps by $0.10/(1 - \rho) = 0.10 \times 1.25 = 0.125$. This is evident in the 3×1 panel of Figure 2, where everything is expressed in percent terms by multiplying by 100. See Appendix B for technical details about how the responses of variables like Y_t , which have no steady state, are computed. The rest of the panels in Figure 2 show how the other model variables respond to the exogenous disturbance in u_t and the consequent movements in $\log Z_t$. The responses are displayed for periods $t = 0, 1, \dots, 99$. This represents a large number of periods, namely 20 years.

Several things are worth emphasizing in these results.

- Note first from the 3,1 panel in Figure 2 that $\log Z_t$ reaches its new steady state growth path almost immediately. By contrast, it takes roughly 10 years for $\log A_t$ to reach its new steady state growth path. The slow response of A_t to a shock in Z_t can also be seen in the behavior of A_t/Z_t , which declines initially and then converges back to its old steady from below (see panel 4,3). The fact that A_t is so slow to react to perturbations in Z_t is why the response of the whole system to a statistical innovation in $\log Z_t$ is dragged out over time. Put differently, the model dynamics are characterized by substantial *internal* propagation.
- Consistent with the previous observation, note how consumption, output, the capital stock all require over 20 years to converge to their new steady state growth path after the shock. The fact that this model emphasizes such low frequencies of oscillation is the

reason Comin and Diego think of their model as a model of ‘intermediate fluctuations’, rather than business cycle fluctuations.

- The new steady state growth path for aggregate quantities corresponds to the growth path in Z_t that is 12.5 percent higher than the growth path of Z_t would have been on if a shock had not occurred. Because the aggregate quantities are scaled by z_t and not Z_t , to determine precisely how much higher their new growth path is one needs to determine how much higher the new steady state growth path of z_t is. For this, recall (4) and the fact that A_t is proportional to Z_t in steady state. As a result, the 12.5 percent eventual rise in Z_t in the steady state translates into a $(\theta - 1) / (1 - \rho) \times 12.5 = 5.86$ percent rise in the aggregate quantities.
- The 3,2 panel indicates that the amount of resources devoted to individual un-developed ideas falls with the expansion, although total investment (panel 4,1) in development goes up. The eventual fall is 6.6 percent. The fact that h_t must eventually fall is a consequence of the fact that h_t must be scaled by multiplication by A_t/z_t , not $1/z_t$ and is discussed in the text.
- Note how hours worked and investment in capital initially fall and then rise. The intuition is that these decisions are made by the household, and for the household the movement in A_t and the consequent movements in the capital rental rate and wage rate are the same as if the household were living in a real business cycle economy. From this perspective, the shock is primarily a signal that A_t will increase in the *future*. Because of this, the household has an incentive to take a sort of ‘vacation’ now: raise consumption and reduce work effort. This manifests itself in the relatively strong immediate rise in consumption and the fall in employment and output that we see in the figures. Of course, it virtually follows that investment must fall, as it does. Although investment in development rises, this takes a very small amount of resources, so that total investment (see the 5,2 panel) falls. Not surprisingly, the physical stock of capital falls for a while, before eventually rising.
- The value of the stock market rises. Significantly, this is not the case for each of the three components. The value of already implemented ideas actually drops for the first 6 years (see panel 2,2). The value of known, but not yet implemented ideas (panel 2,3) drops (10 percent!) in period 0, but then rises enormously relative to its unshocked path. Presumably, the reason that the value of ideas under development rises although the value of known ideas falls for a while is that ideas under development incorporate the value of known ideas relatively far into the future. Recall that the average time it takes to convert an idea into something practical is 5 years (20 quarters) in the model. According to these results, how the stock market responds to the news that A_t and Z_t rise in the future depends very much on how we interpret the stock market in the data. If we think of the real-world stock market in the way we have done in this note, then the stock market jumps in response to an innovation in Z_t . However, if we think that the real-world stock market underrepresents our plants with no-yet-discovered ideas,

then the stock market falls. (This would be even more so if it also underrepresents our plants with ideas that are under development.) For example, panel 5,5 displays the response of the stock market if we include only currently active plants and plants under development. In this case, the model predicts that the stock market is down 8.6 percent in the quarter of the shock, down 1.3 percent in the quarter afterward, before being up 0.2 percent in the quarter thereafter (these numbers are relative to the unshocked path). How to construct a version of the model stock market that is comparable to the real-world stock market is an important question. This is so in part because there is interest in understanding the stock market. Another reason is that Comin and Gertler in later work with Santacreu (2008) place emphasis on their theory's implication that the stock market rises in response to future good news, something that is hard to get in more standard models (see Christiano, Ilut, Rostagno and Motto (2008).)

4 Concluding Remarks

We have explored the properties of a model suggested by Comin and Gertler. The analysis raises several questions that would be interesting to investigate in further research. For example, we assumed that new ideas do not affect the usefulness of incumbent ideas. Once a production process in the model is in place, it continues to be used forever. Under Schumpeter's notion of 'creative destruction', new ideas displace old ideas.⁶ In contrast, Jacob Schmookler suggested that new ideas enhance the productivity of old ideas. These considerations raise two questions: which model of new ideas best captures the data? What would be the best way to integrate one or the other way of thinking about new ideas into a model?

A second class of questions concerns how best to define the stock market in the model, so that it matches up with real-world data on the stock market. In computing the value of the stock market in the model, is it appropriate to include the value of not-yet-discovered new ideas? Our working assumption in this handout follows Comin, Gertler and Santacreu (2008) in assuming that the answer is 'yes'. That is, we assume that plants associated with not-yet-discovered ideas are fully represented in the stock market. Also, it is implicit in the model of this handout that the full value of new ideas is appropriated by some economic entity. It may well be, as in Schumpeter (1942) and in the work on creative destruction by

⁶The idea is best expressed in Schumpeter's (1942) own words: "...the contents of the laborer's budget, say from 1760 to 1940, did not simply grow on unchanging lines but they underwent a process of qualitative change. Similarly, the history of the productive apparatus of a typical farm, from the beginnings of the rationalization of crop rotation, plowing and fattening to the mechanized thing of today—linking up with elevators and railroads—is a history of revolutions. So is the history of the productive apparatus of the iron and steel industry from the charcoal furnace to our own type of furnace, or the history of the apparatus of power production from the overshot water wheel to the modern power plant, or the history of transportation from the mailcoach to the airplane. The opening up of new markets, foreign or domestic, and the organizational development from the craft shop and factory to such concerns as U.S. Steel illustrate the same process of industrial mutation—if I may use that biological term—that incessantly revolutionizes the economic structure from within, incessantly destroying the old one, incessantly creating a new one. This process of Creative Destruction is the essential fact about capitalism."

Shleifer (1986), that new ideas are the property of their discoverer for only a little while, after which they become free to everyone.⁷

We may also ask how the considerations raised here work if they were integrated into a monetary model. Initial important steps in this direction have been taken by Comin, Gertler and Santacreu (2008). One might also investigate, for example, what would happen if funding for the development of new ideas required financing, and that this financing involved various kinds of frictions. It seems likely that ‘agency frictions’ would be important. These are the type of frictions that arise in settings in which there is a principal (i.e., the person who advances the funds required to purchase the resources, h_t , needed for development) and an agent (the manager of the plants associated with not-yet-developed ideas). To see why agency problems might be important, recall that in the model described here development resources only turn an idea into something practical with some probability. Agency problems arise if the plant manager has an incentive to misuse development funds. This would be the case if it is difficult to determine whether the failure of an idea to become practical reflects the difficulties inherent in making an idea practical or whether it reflects that funds have not been allocated as they should have been. For example, the plant manager may apply too little effort, or he/she may even abscond with part or all of the development funds. The frictions that arise in this type of environment reflect the effects of the various contractual and other features that are devised to minimize agency problems.

The answers to the questions raised in these concluding remarks are likely to lead to models with substantially different implications from the ones discussed in this handout. The exploration of these implications would be a fun set of research projects.

⁷Shleifer assumes that a new idea (in the form of an improved way to produce output) can be monopolized by its discoverer for a finite amount of time, after which the idea is freely available everywhere. In this sense, a new idea generates fewer profits than it does in the model of this note. Shleifer captures the transformation of the discoverer of an idea from monopoly to competition by assuming there is a ‘competitive fringe’ of producers surrounding the monopolist who threaten to enter, but who do not actually enter, production. When the knowledge of the discovery of a more efficient way to produce becomes freely available, the competitive fringe force the monopolist to set prices and quantities the way a competitive firm would. As in the previous footnote, Schumpeter (1942) best described how it is that a ‘monopolist’ could be forced to behave competitively, even if there are no other firms active in the market: "It is hardly necessary to point out that competition...acts not only when in being but also when it is merely an ever-present threat. It disciplines before it attacks. The businessman feels himself to be in a competitive situation even if he is alone in his field or if, though not alone, he holds a position such that investigating government experts fail to see any effective competition between him and any other firms in the same or a neighboring field and in consequence conclude that his talk, under examination, about his competitive sorrows is all make-believe. In many cases, though not in all, this will in the long run enforce behavior very similar to the perfectly competitive pattern." These considerations suggest that one way to model the process of creative destruction may be to follow the lead of Shleifer (1986).

References

- [1] Comin, Diego and Mark Gertler, 2006, 'Medium-Term Business Cycles,' *American Economic Review*, vol. 96(3), pages 523-551, June.
- [2] Comin, Diego, Mark Gertler and Ana Maria Santacreu, 2008, 'Innovations in Growth Potential as Sources of Output and Asset Price Fluctuations,' manuscript, Harvard Business School.
- [3] Christiano, Lawrence J., Cosmin Ilut, Massimo Rostagno and Roberto Motto, 2008, 'Signals: Implications for Business Cycles and Monetary Policy,' manuscript, Northwestern University (see <http://www.faculty.econ.northwestern.edu/faculty/christiano/research/ECB/sed2008.ppt> and <http://www.faculty.econ.northwestern.edu/faculty/christiano/research/ECB/mirage.htm>).
- [4] Prescott, Edward, 1998, 'Needed: A Theory of Total Factor Productivity,' *International Economic Review*, vol. 39, no. 3, pp. 525-551.
- [5] Schumpeter, Joseph A., 1942, *Capitalism, Socialism and Democracy*.
- [6] Shleifer, Andre, 1986, 'Implementation Cycles,' *Journal of Political Economy*. For a class handout on this, see <http://faculty.econ.northwestern.edu/faculty/christiano/d11/d1101/implement.pdf>.

5 Appendix A: Scaling of the Variables

Following is a summary of the scaling conventions used in the paper:

$$\begin{aligned}
k_{t+1} &= \frac{K_{t+1}}{z_t}, \quad y_t = \frac{Y_t}{z_t}, \quad c_t = \frac{C_t}{z_t}, \quad g_t = \frac{G_t}{z_t}, \quad u_{c,t}^z = z_t u_{c,t}, \quad w_t = \frac{\tilde{w}_t}{z_t}, \\
\mu_{z,t} &= \left(\frac{A_t}{A_{t-1}} \right)^{\frac{\vartheta-1}{1-\alpha}} = \left(\frac{a_t Z_t}{a_{t-1} Z_{t-1}} \right)^{\frac{\vartheta-1}{1-\alpha}} = \left(\frac{a_t \exp(\chi_{t-1})}{a_{t-1}} \right)^{\frac{\vartheta-1}{1-\alpha}} \\
o_{z,t} &= \frac{o_t}{z_t}, \quad z_t = A_t^{\frac{\vartheta-1}{1-\alpha}}, \quad a_t = \frac{A_t}{Z_t}, \\
j_{A,t} &= A_t \frac{J_t}{z_t}, \quad h_{A,t} = \frac{A_t h_t}{z_t} \\
v_{A,t} &= \frac{V_t}{z_t} A_t, \quad \Pi_{At}(j) = \frac{\Pi_t(j)}{z_t} A_t \\
s_t &= \tilde{s}_t A_t^{1-\theta}
\end{aligned}$$

6 Appendix B: Impulse Response Functions

Consider the impulse response to a shock in period 0 of a variable like C_t :

$$\begin{aligned}
C_t &= c_t z_t \\
&= c_t \mu_{z,t} \cdots \mu_{z,1} z_0
\end{aligned}$$

To compute the log deviation from the unshocked path, \bar{C}_t , we have

$$\begin{aligned}
\log \frac{C_t}{\bar{C}_t} &= \log \frac{c_t}{c} + \log \frac{\mu_{z,t} \cdots \mu_{z,1}}{\mu_z^t} = \log \frac{c_t}{c} + \sum_{j=0}^{t-1} \log \frac{\mu_{z,t-j}}{\mu_z} \\
&\simeq \frac{c_t - c}{c} + \sum_{j=0}^{t-1} \frac{\mu_{z,t-j} - \mu_z}{\mu_z}.
\end{aligned}$$

Consider the impulse response function of total resources devoted to making known ideas useful, $h_t(Z_t - A_t)$:

$$\begin{aligned}
\log \frac{h_t(Z_t - A_t)}{\bar{h}_t(\bar{Z}_t - \bar{A}_t)} &= \log \frac{h_{At} z_t \left(\frac{1}{a_t} - 1 \right)}{h_A \bar{z}_t \left(\frac{1}{a} - 1 \right)} \\
&= \log \frac{h_{At}}{h_A} + \log \frac{z_t}{\bar{z}_t} + \log \frac{\left(\frac{1}{a_t} - 1 \right)}{\left(\frac{1}{a} - 1 \right)},
\end{aligned}$$

where

$$\begin{aligned}
\log \frac{\left(\frac{1}{a_t} - 1\right)}{\left(\frac{1}{a} - 1\right)} &= \log \frac{(1 - a_t)}{(1 - a)} + \log \frac{a}{a_t} \\
&\simeq \frac{a - a_t}{1 - a} - \frac{a_t - a}{a} = -(a_t - a) \left[\frac{1}{a} + \frac{1}{1 - a} \right] \\
&= -\frac{a_t - a}{a(1 - a)}
\end{aligned}$$

We conclude:

$$\log \frac{h_t(Z_t - A_t)}{\bar{h}_t(\bar{Z}_t - \bar{A}_t)} \simeq \frac{h_{A_t} - h_A}{h_A} + \sum_{j=0}^{t-1} \frac{\mu_{z,t-j} - \mu_z}{\mu_z} - \frac{a_t - a}{a(1 - a)}.$$

Now consider the response of A_t . Its value relative to the unperturbed path, \bar{A}_t , is

$$\frac{A_t}{\bar{A}_t} = \frac{a_t Z_t}{a \bar{Z}_t} = \begin{cases} \frac{a_t \exp(\chi_{t-1} + \chi_{t-2} + \dots + \chi_0) Z_0}{a \exp(t\chi) Z_0} & t > 0 \\ \frac{a_t}{a} & t = 0 \end{cases}.$$

Then,

$$\begin{aligned}
\log \left(\frac{A_t}{\bar{A}_t} \right) &= \begin{cases} \log \frac{a_t}{a} + \sum_{j=1}^t (\chi_{t-j} - \chi) & t > 0 \\ \log \frac{a_t}{a} & t = 0 \end{cases} \\
&\simeq \begin{cases} \frac{a_t - a}{a} + \sum_{j=1}^t (\chi_{t-j} - \chi) & t > 0 \\ \frac{a_0 - a}{a} & t = 0 \end{cases}.
\end{aligned}$$

7 Appendix C: Summary of Equilibrium Conditions

We have 14 unknowns: $\{r_t, w_t, \mu_{z,t}, R_{t+1}, k_t, L_t, y_t, c_t, a_t, h_{A,t}, o_{z,t}, \chi_t, j_{A,t}, v_{A,t}\}$, and 14 equations:

$$r_t = \frac{1}{\theta} \alpha \epsilon_t \left(\frac{k_t}{L_t} \frac{1}{\mu_{z,t}} \right)^{\alpha-1} \quad (19)$$

$$w_t = \frac{1}{\theta} (1 - \alpha) \epsilon_t \left(\frac{k_t}{L_t} \frac{1}{\mu_{z,t}} \right)^{\alpha} \quad (20)$$

$$\mu_{z,t} = \left(\frac{a_t \exp(\chi_{t-1})}{a_{t-1}} \right)^{\frac{\theta-1}{1-\alpha}} \quad (21)$$

$$y_t = \epsilon_t \left(\frac{k_t}{\mu_{z,t}} \right)^{\alpha} L_t^{1-\alpha} \quad (22)$$

$$v_{A,t} = \frac{\theta - 1}{\theta} y_t + \beta E_t \frac{u_{c,t+1}^z}{u_{c,t}^z} \frac{a_t}{a_{t+1} \exp(\chi_t)} v_{A,t+1} \quad (23)$$

$$\chi_t = (1 - \rho) \bar{\chi} + \rho \chi_{t-1} + u_t \quad (24)$$

$$a_{t+1} = \frac{\lambda(h_{A,t}/o_{z,t})}{\exp(\chi_t)} + \frac{1 - \lambda(h_{A,t}/o_{z,t})}{\exp(\chi_t)} a_t \quad (25)$$

$$o_{z,t} = \left(\frac{a_{t-1}}{a_t} \frac{1}{\exp(\chi_t)} \right)^{\frac{\theta-1}{1-\alpha}} b k_t \quad (26)$$

$$1 = \beta E_t \frac{u_{c,t+1}^z}{u_{c,t}^z} \lambda' \left(\frac{h_{A,t}}{o_{z,t}} \right) \frac{a_t}{o_{z,t} a_{t+1} \exp(\chi_t)} (v_{A,t+1} - j_{A,t+1}) \quad (27)$$

$$j_{A,t} = -h_{A,t} + \beta E_t \frac{u_{c,t+1}^z a_t}{u_{c,t}^z a_{t+1} \exp(\chi_t)} \left[\lambda \left(\frac{h_{A,t}}{o_{z,t}} \right) v_{A,t+1} + \left(1 - \lambda \left(\frac{h_{A,t}}{o_{z,t}} \right) \right) j_{A,t+1} \right] \quad (28)$$

$$w_t = \psi L_t^{\zeta} c_t \quad (29)$$

$$1 = \beta E_t \frac{c_t}{c_{t+1}} \frac{1}{\mu_{z,t+1}} (1 + r_{t+1} - \delta) \quad (30)$$

$$R_{t+1} = \left(\beta E_t \frac{c_t}{c_{t+1}} \frac{1}{\mu_{z,t+1}} \right)^{-1} \quad (31)$$

$$c_t + k_{t+1} - (1 - \delta) k_t / \mu_{z,t} + \frac{h_{A,t}}{a_t} (1 - a_t) + g_t = y_t. \quad (32)$$

8 Appendix D: Computation of Steady State

In steady state, the system of equations is as follows:

$$r = \frac{1}{\theta} \alpha \left(\frac{k}{L} \frac{1}{\mu_z} \right)^{\alpha-1} \quad (33)$$

$$w = \frac{1}{\theta} (1 - \alpha) \left(\frac{k}{L} \frac{1}{\mu_z} \right)^{\alpha} \quad (34)$$

$$\mu_z = [\exp(\chi)]^{\frac{\theta-1}{1-\alpha}} \quad (35)$$

$$y = \epsilon \left(\frac{k}{\mu_z} \right)^{\alpha} L^{1-\alpha} \quad (36)$$

$$v_A = \frac{\frac{\theta-1}{\theta} y}{1 - \frac{\beta}{\exp(\chi)}} \quad (37)$$

$$\chi = \bar{\chi} \quad (38)$$

$$a = \frac{\lambda(h_A/o_z)}{\exp(\chi) - [1 - \lambda(h_A/o_z)]} \quad (39)$$

$$o_z = \left(\frac{1}{\exp(\chi)} \right)^{\frac{\theta-1}{1-\alpha}} bk \quad (40)$$

$$1 = \beta \lambda' \left(\frac{h_A}{o_z} \right) \frac{1}{o_z \exp(\chi)} (v_A - j_A) \quad (41)$$

$$j_A = -h_A + \beta \frac{1}{\exp(\chi)} \left[\lambda \left(\frac{h_A}{o_z} \right) v_A + \left(1 - \lambda \left(\frac{h_A}{o_z} \right) \right) j_A \right] \quad (42)$$

$$w = \psi L^{\zeta} c \quad (43)$$

$$1 = \beta \frac{1}{\mu_z} (1 + r - \delta) \quad (44)$$

$$R = \left(\beta \frac{1}{\mu_z} \right)^{-1} \quad (45)$$

$$c + k - (1 - \delta)k/\mu_z + \frac{h_A}{a} (1 - a) + g = y. \quad (46)$$

To solve for capital-labor ratio (k/L), we substitute (33) into (44):

$$\frac{k}{L} = \left[\frac{\theta}{\alpha \epsilon} \left(\frac{\mu_z}{\beta} + \delta - 1 \right) \right]^{\frac{1}{\alpha-1}} \mu_z$$

where

$$\mu_z = [\exp(\bar{\chi})]^{\frac{\theta-1}{1-\alpha}}$$

by (35) and (38). The last two equations, together with (33), (34) and (35) pin down $r, w, R, \mu_z, k/L$. Fix a value for L . Solve for y and v_A using (36) and (37), respectively. Solve

for c from (43). Solve for

$$\frac{h_A}{a}(1 - a) = \kappa,$$

say, using $k = (k/L)L$, $g = \eta_g y$, and (46). Then, using (39):

$$\kappa = h_A \left(\frac{\exp(\chi) - [1 - \lambda(h_A/o_z)]}{\lambda(h_A/o_z)} - 1 \right),$$

is known. But,

$$\lambda = \left(\frac{h_A}{o_z} \right)^\sigma, \quad \sigma > 0$$

so that

$$h_A = o_z \lambda^{\frac{1}{\sigma}}.$$

Substituting,

$$o_z \lambda^{\frac{1}{\sigma}-1} [\exp(\chi) - 1] = \kappa.$$

Then,

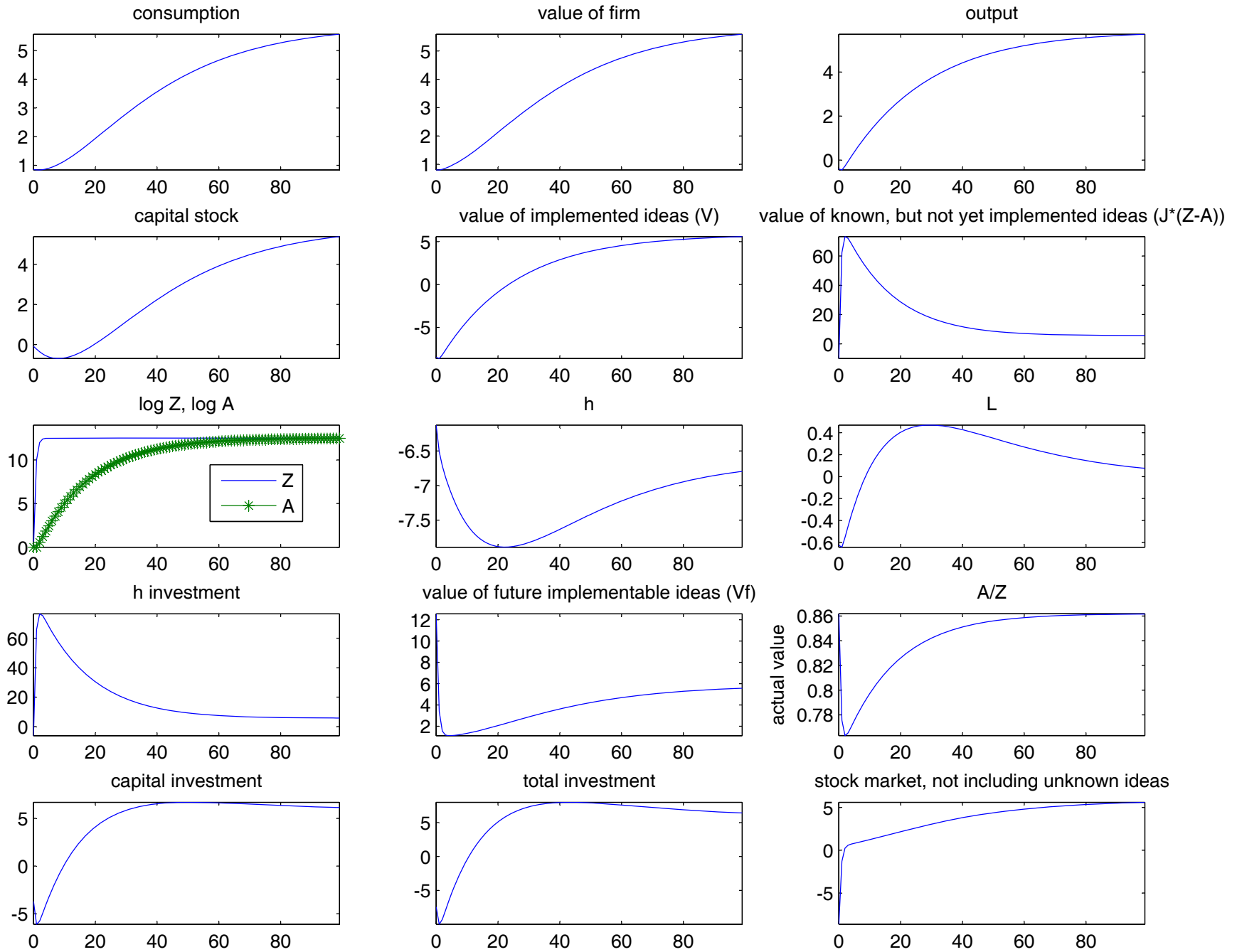
$$\lambda = \left(\frac{\kappa}{o_z [\exp(\chi) - 1]} \right)^{\frac{\sigma}{1-\sigma}}.$$

Next, solve for j_A after substituting (37) into (42):

$$j_A = \frac{\lambda \frac{\beta}{\exp(\bar{\chi}) - \beta} \frac{\theta - 1}{\theta} y - h_A}{1 - \frac{\beta}{\exp(\bar{\chi})} [1 - \lambda]}.$$

Finally, adjust L until (41) is satisfied.

Figure 2: Dynamic Response to Innovation in Ideas



Note: with the one exception that is indicated, all curves represent percent deviations from the unshocked growth path