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Econ 411, Fall 2008

MIDTERM EXAM

There are four questions. The total number of possible points is 100, and the number of points per question is given in parentheses. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have 2 hours. Good luck!

1. (35) The typical household can engage in two types of activities: producing current output and studying at home. Although time spent on studying at home sacrifices current production, it augments future output by increasing the household's future stock of human capital, k_{t+1}. The household has one unit of time available to split between home study and current production. Any given amount of human capital accumulation, k_{t+1}/k_t, leaves an amount of time, h_t, left over for producing current output, where $h_t = \phi(k_{t+1}/k_t)$. Here, ϕ is strictly decreasing, strictly concave, and continuously differentiable, with

$$\phi(1-\delta) = 1 \text{ for some } \delta \in (0,1),$$
$$\phi(1+\lambda) = 0 \text{ for some } \lambda > 0,$$

where

$$\beta\,(1+\lambda)^{\alpha\sigma} < 1.$$

The variable, h_t, must satisfy $0 \leq h_t \leq 1$. A household's effective labor input into production is the product of its time and human capital: $h_t k_t$. Total output is related to effective labor input by

$$f(h_t k_t) = (h_t k_t)^{\alpha},\ \alpha \in (0,1).$$

The resource constraint for this economy is

$$c_t \leq f(h_t k_t),$$

and the initial level of human capital, k_0, is given. The utility value of a given sequence of consumption, c_t, is given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t),\ \text{where } u(c_t) = c_t^{\sigma}/\sigma,\ \sigma < 1.$$

(a) Write out the sequence problem. Let $v(k_0)$ denote the present discounted value of utility, conditional on the initial stock of human capital, associated with the efficient allocations.

(b) Show that $v(k) = Ak^{\sigma\alpha}$.

(c) Show that $-\infty < A < \infty$. Show how the assumed bound on λ is used in establishing this result.

(d) Show that A can be expressed as the fixed point of a mapping from a bounded subset of the real line into itself. Show that that mapping is a contraction mapping, so that the solution is unique. Display an algorithm that can be used to find the solution.

(e) Explain why the efficient allocations are attained by a policy function of the form, $g(k) = \theta k$ for some $(1-\delta) \leq \theta \leq (1+\lambda)$.

2. (25) Consider the following sequence problem:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to:

$$c_t + k_{t+1} - (1-\delta)k_t + g \leq f(k_t),$$
$$c_t, k_{t+1} \geq 0, \ \ u, f \text{ strictly concave, increasing, } g > 0,$$
$$f'(k) \to 0 \text{ as } k \to \infty.$$

Note that this resource constraint differs from the type of constraint considered in class because of the presence of $g > 0$. We can think of this as a fixed level of government spending.

(a) Show that there is some $\underline{k} > 0$ such that if ever $k_t < \underline{k}$, then $c_{t+j} \geq 0$ for all $j \geq 0$ is technologically infeasible.

(b) Show that there is some $\overline{k} > \underline{k}$ such that if $k_t \leq \overline{k}$, then technological feasibility requires $k_{t+j} \leq \overline{k}$ for all $j \geq 0$.

3. (20) Consider a household which solves the following problem:

$$v(k, r, w) = \max_{c, l \in B(k,r,w)} u(c, l),$$

where $u : \Re^2_+ \rightarrow \Re$ is a strictly concave, twice continuously differentiable, strictly increasing function in its two arguments: consumption, c, and leisure, l. The constraints the household must obey in selecting c, l are summarized by B:

$$B(k, r, w) = \{c, l : 0 \leq c \leq rk + w(1 - l),\ 0 \leq l \leq 1\}.$$

Here, $r > 0$ is the market rental rate on capital and $w > 0$ is the market wage rate, neither of which the household can control. Also, $k > 0$ is the household's stock of capital. Prove that the derivative of v with respect to k exists, and display a formula for it. If you make use of a theorem to help prove your result, be sure to state it clearly.

4. (20) Consider the canonical model with the following sequence representation:

$$v(x_0) = \max_{x_{t+1} \in \Gamma(x_t)} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}),\ x_t \in X,$$

where $F : A \rightarrow R$, $A = \{x, x' : x \in X,\ x' \in \Gamma(x)\}$, $\Gamma : X \rightarrow X$. Suppose A1: X is convex; Γ is non-empty, compact, continuous, A2: F is bounded, continuous and $0 < \beta < 1$. Also, A3: F is strictly increasing in its first argument for each fixed value of its second argument, A4: Γ is monotone. Prove formally that v is strictly increasing. You may simply state results which guarantee v exists and is unique. Be sure to point out the role in the argument of each of the assumptions, A1-A4.