Economics 411
Fall, 2009
Final exam, December 7
Christiano

FINAL EXAM

Each of the four questions is worth 25 points. Allocate your time accordingly. If a question seems ambiguous, state why, sharpen it up and answer the revised question. Good luck!

1. The typical household can allocate its one unit of available time in two ways. It can spend $h_t$ units of time, $0 \leq h_t \leq 1$, producing goods and it spends the rest of its time studying at home. Although time spent studying at home sacrifices current production, studying augments future output by increasing the household's future stock of human capital, $k_{t+1}$. There are technologically imposed upper and lower limits on human accumulation:

$$1 - \delta \leq \frac{k_{t+1}}{k_t} \leq 1 + \lambda,$$

for some $\delta \in (0, 1)$ and $\lambda > 0$. To simplify analysis, we impose the following boundedness condition:

$$\beta (1 + \lambda)^{\alpha\sigma} < 1.$$

The time left for producing current output, $h_t$, given the amount of human capital accumulation is given by the function, $h_t = \phi(k_{t+1}/k_t)$. Here, $\phi$ is strictly decreasing, strictly concave, and continuously differentiable, with

$$\phi(1 - \delta) = 1, \ \phi(1 + \lambda) = 0.$$

A household's effective labor input into production is the product of its time and human capital: $h_t k_t$. Total output is related to effective labor input by

$$f(h_t k_t) = (h_t k_t)^{\alpha}, \ \alpha \in (0, 1).$$

The resource constraint for this economy is

$$c_t \leq f(h_t k_t),$$

and the initial level of human capital, $k_0$, is given. The utility value of a given sequence of consumption, $c_t$, is given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \text{ where } u(c_t) = c_t^{\sigma}/\sigma, \ 0 < \sigma < 1.$$

If it is useful, you may assume, without proof, that the optimal growth rate of human capital in this economy is strictly interior to the set $[(1-\delta), 1+\lambda]$. If you use this assumption, be sure to say so explicitly.

(a) Write out the sequence problem. Let $v(k_0)$ denote the present discounted value of utility, conditional on the initial stock of human capital, associated with the efficient allocations.

(b) Show that $v(k) = Ak^{\sigma\alpha}$.

(c) Show that $-\infty < A < \infty$. Show how the assumed bound on $\lambda$ is used to establish this result.

(d) Show that $A$ can be expressed as the fixed point of a mapping from a bounded subset of the real line into itself. Show that that mapping is a contraction mapping, so that the solution is unique. Display an algorithm that can be used to find the solution.

(e) Explain why the efficient allocations are attained by a policy function of the form, $g(k) = \theta k$ for some $(1-\delta) \leq \theta \leq (1+\lambda)$.

(f) Show that if there are two economies that are identical, except that one has a higher value of $\beta$, the high $\beta$ economy will grow faster. What is the intuition for this result?

2. Consider a model in which a final good is produced using intermediate goods. The final good, $Y$, is produced by a competitive, representative firm using the following homogeneous technology:

$$Y = \exp \int_0^1 [\log y_j] \, dj.$$

The firm maximizes profits:

$$Y - \int_0^1 p_j y_j dj,$$

taking $p_j$ as given. Here, the price of the final good has been normalized at unity. The $j^{th}$ intermediate good is produced by a monopolist using the following technology:

$$\begin{aligned} y_j &= \begin{cases} f(k_j, l_j) - \phi & f(k_j, l_j) \geq \phi \\ 0 & \text{otherwise} \end{cases}, \\ f(k_j, l_j) &= k_j^{\alpha} l_j^{1-\alpha}, \ 0 < \alpha < 1. \end{aligned} \tag{1}$$

Here, $k_j$ and $l_j$ denote the quantity of capital and labor, respectively, employed by the monopolist. Thus, if the monopolist is to sell $y_j$ units of goods, they must produce the fixed quantity, $\phi$, first. The monopolist is competitive in the market for labor and capital and takes the rental rate on capital, $r$, and the wage rate, $w$, as given.

(a) Derive the demand curve for the $j^{th}$ intermediate good.

(b) Consider the problem of the $j^{th}$ intermediate good firm.

  i. Consider the firm's cost minimization problem and show that the firm that produces a quantity $y_j$ has total cost

  $$s\left(y_j + \phi\right), \ s = \left(\frac{w}{1-\alpha}\right)^{1-\alpha} \left(\frac{r}{\alpha}\right)^{\alpha}.$$

  Explain in intuitive terms why marginal cost, $s$, is independent of the scale of firm $j$'s production.

  ii. Show that the monopoly price setting problem has no solution. That is, for any finite price-quantity pair on the monopolist's demand curve, profits are always increased by raising $p_j$.

(c) Suppose that there are other potential entrants into the production of the $j^{th}$ intermediate good, and that they have access to the same technology, (1). Explain why this implies that the profits of the intermediate good producer must be zero.

(d) Show the $j^{th}$ intermediate good producer's output can be written

$$y_j = \frac{1}{\mu_j} f(k_j, l_j),$$

where $\mu_j$ is the firm markup, the ratio of price to marginal cost, $\lambda_j$:

$$\mu_j = \frac{p_j}{\lambda_j}.$$

(e) Show that the zero profit condition implies the markup must fall when the firm produces more output. Provide the intuition for this result.

(f) Prove that total income, $rk + wl$, is equal to $Y$, and that capital and labor receive shares, $\alpha$ and $1 - \alpha$, respectively, of this income.

(g) Explain why it is that in equilibrium, final output has the following representation:

$$Y = \frac{1}{\mu} f(k, l),$$

where $l$ is household labor supply, $k$ is the supply of capital by households, and $\mu$ is the markup. Suppose that $k$, $l$ and $Y$ fluctuate in response to shocks originating outside the firm sector. Explain how the framework of this question provides a 'theory of TFP' in that an econometrician living in this economy would estimate that TFP is procyclical. (Hint: TFP is the ratio of $Y$ to the product of capital and labor, each raised to a power corresponding to its share in final output, $Y$.) Explain why an econometrician who is also a real business cycle theorist might conclude that RBC theory is vindicated when it is not.

3. (Dynamic Inefficiency in OG Models). Consider the overlapping generations model in which the utility of the generation born at $t$ is

$$u(c_t^t, c_{t+1}^t) = \log(c_t^t) + \beta \log(c_{t+1}^t).$$

The young supply one unit of labor inelastically in period zero, and earn the competitive wage rate, $w_t$. They use their income to purchase the outstanding stock of capital, and when old they finance their consumption from the earnings of the accumulated capital. Thus, their budget constraint is

$$c_t^t + k_{t+1} \leq w_t, \; c_{t+1}^t \leq r_{t+1} k_{t+1}.$$

Note that capital depreciates completely in one period. Firms are competitive in the output market and hire capital and labor in competitive factor markets where the prices are $r_t$ and $w_t$, respectively. Their production technology is $y = k^\alpha n^{1-\alpha}$, $0 < \alpha < 1$.

(a) Define a sequence of markets equilibrium. Provide expressions for $w_t$ and $r_t$ in terms of $k_t$.

(b) Consider a steady state equilibrium in which the aggregate stock of capital is constant, $k$, the consumption of each period's young is constant, $c^y$, and the consumption of each period's old is a constant, $c^o$. Time starts up in period 0, with the initial old generation owning the (steady state) capital stock, which they sell to the period 0 young.

   i. Display the six equilibrium conditions of the model: the household intertemporal efficiency condition, the two household budget constraints, the two firm efficiency conditions and the resource constraint. At the same time, there are only 5 unknowns in a steady state equilibrium, $c^o$, $c^y$, $r$, $w$, $k$. Prove that the apparent overdetermination of the equilibrium reflects only that there is a redundancy among the six equilibrium conditions (hint: remember Walras' law).

   ii. Derive expressions relating the three household decisions, $k$, $c^y$, $c^o$ to the factor prices and model parameters.

   iii. Explain why the rental rate on capital, $r$, is also the rate of return on capital in this model. Show that the equilibrium rate of return on capital is

$$r = \frac{\alpha}{1-\alpha} \frac{1+\beta}{\beta}.$$

   Provide intuition for this equation. Why is the interest rate infinite if $\beta = 0$? Why is it zero if $\alpha = 0$?

(c) Show that, for parameter values where $r < 1$, the competitive equilibrium is inefficient.

i. Prove the following: given a steady state competitive equilibrium with $r < 1$, it is possible to construct a welfare improving, feasible deviation. The deviation reallocates consumption from the young to the old, and makes each generation better off. (By each generation, I mean the initial old, those born in periods 0, 1, 2, .....).

ii. How might the result be affected if there were a last date in the economy? (In the last date, there are initial young who will not live into old age, and last period old people.)

4. Entrepreneurs have access to a technology for converting capital, $k$, into output

$$\omega k^{\alpha},$$

where $\omega$ is a technology shock drawn independently by each entrepreneur from a distribution with $E\omega = 1$ and cumulative distribution function $F(x) \equiv prob\,[\omega \leq x]$. The realization of $\omega$ is observed by the entrepreneur, and can be seen by a lender only if the lender pays a monitoring cost, $\mu k^{\alpha}$. The depreciation rate on capital is $\delta$, so that after production, entrepreneurs have $(1-\delta)\,k$ units of capital left, which they can sell at a price of unity (the price of output is the numeraire). Thus, after production the entrepreneur who draws $\omega$ has the following resources:

$$y(\omega) = \omega k^{\alpha} + (1-\delta)\,k, \;\; 0 < \delta, \alpha \leq 1,$$

At the beginning of the period, entrepreneurs have no capital, but they do have net worth, $n$, that they can use for purchasing $k$. Suppose there are many entrepreneurs with each possible level of $n$.

Consider an entrepreneur with net worth, $n$, who purchases an amount of capital, $k > n$. The entrepreneur borrows $b \equiv k - n$ at gross rate of interest, $Z$, from a bank. There is a large number of banks that specialize in lending to entrepreneurs with each level of net worth, $n$, and there is free entry into banking. In case the entrepreneur's revenue, $y(\omega)$, falls below the required payment to the bank, $Z(k-n)$, the entrepreneur declares bankruptcy and is monitored. In addition, the bank takes whatever the entrepreneur has. Let $\bar{\omega}$ be defined by

$$\bar{\omega}k^{\alpha} + (1-\delta)\,k = Z\,(k-n)\,.$$

Prior to making loans to entrepreneurs, banks have access to a competitive market in which they can borrow as much or as little as they want, at gross rate of interest, $R$. At the end of the period, when the banks have to repay household loans, the only source of funds available to them is the funds given to them by entrepreneurs. Entrepreneurial utility prior to production is proportional to their expected end-of-period resources, net of bank costs.

(a) Show that the expected profits of an entrepreneur with net worth $n$, interest rate $Z$, and loan amount $k-n$ can be written

$$\left[1-\Gamma\left(\bar{\omega}\right)\right]k^{\alpha},$$

where

$$\Gamma\left(\bar{\omega}\right) = \left[1-F\left(\bar{\omega}\right)\right]\bar{\omega} + \int_0^{\bar{\omega}} \omega dF\left(\omega\right).$$

(b) Suppose that each bank deals with a large, randomly selected set of entrepreneurs. Show that the average revenues, across all loans to entrepreneurs with net worth $n$, is

$$\left[\Gamma\left(\bar{\omega}\right) - \mu F\left(\bar{\omega}\right)\right]k^{\alpha} + (1-\delta)\,k.$$

(c) Display the zero profit condition for the banks that lend to entrepreneurs with net worth, $n$. Also, display on optimization problem whose solution gives the equilibrium contract $(Z,b)$ (You do not have to solve for the equilibrium contract).