Answers, final.
Economics 411
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Christiano

1. 1. The typical household can engage in two types of activities: producing current output and studying at home. Although time spent on studying at home sacrifices current production, it augments future output by increasing the household's future stock of human capital, k_{t+1}. The household has one unit of time available to split between home study and current production. Any given amount of human capital accumulation, k_{t+1}/k_t, leaves an amount of time, h_t, left over for producing current output, where $h_t = \phi(k_{t+1}/k_t)$. Here, ϕ is strictly decreasing, strictly concave, and continuously differentiable, with

$$\phi(1-\delta) = 1 \text{ for some } \delta \in (0,1),$$
$$\phi(1+\lambda) = 0 \text{ for some } \lambda > 0.$$

The variable, h_t, must satisfy $0 \leq h_t \leq 1$. A household's effective labor input into production is the product of its time and human capital: $h_t k_t$. Total output is related to effective labor input by

$$f(h_t k_t) = (h_t k_t)^{\alpha},\ \alpha \in (0,1).$$

The resource constraint for this economy is

$$c_t \leq f(h_t k_t),$$

and the initial level of human capital, k_0, is given. The utility value of a given sequence of consumption, c_t, is given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t),\ \text{where } u(c_t) = c_t^{\sigma}/\sigma,\ \sigma < 0.$$

a, b, c (10) Express the planning problem for this economy as a sequence problem (SP). Write out the associated functional equation (FE).

$$v(k_0) = \max \sum_{t=0}^{\infty} \beta^t \frac{c_t^\sigma}{\sigma},$$

subject to

$$c_t = f(h_t k_t), \;\; h_t = \phi(k_{t+1}/k_t)$$

so,

$$v(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{(\phi(k_{t+1}/k_t)k_t)^{\alpha\sigma}}{\sigma} = \max_{\{\lambda_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t k_t^{\alpha\sigma} \frac{\phi(\lambda_t)^{\alpha\sigma}}{\sigma},$$

where

$$\lambda_t = k_{t+1}/k_t.$$

But,

$$k_t = k_0 \Pi_{j=0}^{t-1} \lambda_j,$$

so,

$$\begin{aligned} v(k_0) &= \max_{\{\lambda_t\}_{t=0}^{\infty}} k_0^{\alpha\sigma} \sum_{t=0}^{\infty} \beta^t \left[\Pi_{j=0}^{t-1} \lambda_j^{\alpha\sigma}\right] \frac{\phi(\lambda_t)^{\alpha\sigma}}{\sigma} \\ &= k_0^{\alpha\sigma} \max_{\{1-\delta \le \lambda_t \le 1+\lambda\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\Pi_{j=0}^{t-1} \lambda_j^{\alpha\sigma}\right] \frac{\phi(\lambda_t)^{\alpha\sigma}}{\sigma} \\ &= A k_0^{\alpha\sigma}, \end{aligned}$$

where

$$A = \sum_{t=0}^{\infty} \beta^t \left[\Pi_{j=0}^{t-1} \lambda_j^{\alpha\sigma}\right] \frac{\phi(\lambda_t)^{\alpha\sigma}}{\sigma}.$$

To establish $-\infty < A < \infty$, proceed as follows. That $A > -\infty$ follows from the fact that the objective is non-negative. Now, we establish that $A < \infty$. Then,

$$\begin{aligned} A &= \max_{\{1-\delta \le \lambda_t \le 1+\lambda\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\Pi_{j=0}^{t-1} \lambda_j^{\alpha\sigma}\right] \frac{\phi(\lambda_t)^{\alpha\sigma}}{\sigma} < \sum_{t=0}^{\infty} \left\{ \max_{\{1-\delta \le \lambda_t \le 1+\lambda\}_{t=0}^{\infty}} \beta^t \left[\Pi_{j=0}^{t-1} \lambda_j^{\alpha\sigma}\right] \right\} \left\{ \max_{\{1-\delta \le \lambda_t \le 1+\lambda\}_{t=}^{\infty}} \right. \\ &= \sum_{t=0}^{\infty} \left\{\beta \left(1+\lambda\right)^{\alpha\sigma}\right\}^t \left\{\frac{\phi(1-\delta)^{\alpha\sigma}}{\sigma}\right\} < \infty, \end{aligned}$$

given the boundedness condition.

Writing out the expression for A carefully, we find:

$$A = \max_{\lambda} \frac{\phi(\lambda)^{\alpha\sigma}}{\sigma} + \beta\lambda^{\alpha\sigma} A$$

Define the following mapping, $T\left(w\right)$:

$$T\left(w\right) = \max_{\lambda} \frac{\phi(\lambda)^{\alpha\sigma}}{\sigma} + \beta\lambda^{\alpha\sigma} w.$$

To verify that this mapping has a unique solution, verify that Blackwell's sufficient conditions are satisfied. Monotonicity requires:

$$T\left(w\right) \leq T\left(v\right), \text{ if } w < v.$$

To verify this, let

$$\lambda_w \equiv \arg\max_{\lambda} \frac{\phi(\lambda)^{\alpha\sigma}}{\sigma} + \beta\lambda^{\alpha\sigma} w,$$

so that

$$\begin{aligned} T\left(w\right) &= \frac{\phi(\lambda_w)^{\alpha\sigma}}{\sigma} + \beta\lambda_w^{\alpha\sigma} w \\ &\overbrace{<}^{\text{by } w<v} \frac{\phi(\lambda_w)^{\alpha\sigma}}{\sigma} + \beta\lambda_w^{\alpha\sigma} v \\ &\overbrace{\leq}^{\text{by optimality}} T\left(v\right). \end{aligned}$$

Discounting requires that

$$T\left(w+g\right) \leq T\left(w\right) + \varepsilon g,$$

where $\varepsilon \in (0,1)$ and $g \geq 0$ is a scalar.

$$\begin{aligned} T\left(w+g\right) &= \max_{\lambda} \frac{\phi(\lambda)^{\alpha\sigma}}{\sigma} + \beta\lambda^{\alpha\sigma}\left[w+g\right] \\ &\leq \max_{\lambda} \left[\frac{\phi(\lambda)^{\alpha\sigma}}{\sigma} + \beta\lambda^{\alpha\sigma} w\right] + \max_{\lambda} \beta\lambda^{\alpha\sigma} g \\ &= T\left(w\right) + \max_{\lambda} \beta\lambda^{\alpha\sigma} g. \end{aligned}$$

Recall that $1-\delta \leq \lambda_t \leq 1+\lambda$. The solution to the above maximization problem is the largest possible value of λ which is $1+\lambda$. In this case,

$$T\left(w+g\right) \leq T\left(w\right)+\beta\left(1+\lambda\right)^{\alpha\sigma} g.$$

But, $0<\beta\left(1+\lambda\right)^{\alpha\sigma}<1$. Discounting is established by setting $\varepsilon=\beta\left(1+\lambda\right)^{\alpha\sigma}$.

Thus, T is a contraction and it has a unique fixed point, arrived at by

$$A=\lim_{j\rightarrow\infty} T^{j} w,$$

for any initial w. The optimal value of λ, θ, is

$$\theta=\arg\max_{\{1-\delta\leq\lambda\leq 1+\lambda\}} \frac{\phi(\lambda)^{\alpha\sigma}}{\sigma}+\beta\lambda^{\alpha\sigma} A$$

But, θ is defined as k_{t+1}/k_t, so that the solution is a policy rule,

$$g\left(k_t\right)=\theta k_t.$$

f. The graph of $\phi(\lambda)^{\alpha\sigma}/\sigma$ against λ is downward sloped. The graph of $\beta\lambda^{\alpha\sigma}A$ against λ is positively sloped. Assuming the efficient growth rate is interior, these two curves intersect in the interior of the set that restricts λ. Only the positively sloped graph is a function of β. That graph shifts to the right with an increase in β, and so the intersection of the two curves (assuming the equilibrium is interior) shifts to the right, to a higher value of λ.

2 The optimization problem is

$$\max Y-\int_0^1 p_j y_j dj.$$

with first order conditions:

$$\frac{Y}{y_j}=p_j. \tag{1}$$

Marginal cost for the j^{th} intermediate good firm is

$$s=\left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{w}{1-\alpha}\right)^{1-\alpha}.$$

This is obtained by studying its cost minimization problem:

$$\min_{l_j,k_j} wl_j + rk_j + s\left[y_j - f\left(k_j, l_j\right) - \phi\right].$$

This problem leads to the following first order necessary condition for an interior optimum:

$$\begin{aligned} w &= s\left(1-\alpha\right)\left(\frac{k_j}{l_j}\right)^{\alpha} \\ r &= s\alpha\left(\frac{k_j}{l_j}\right)^{\alpha-1} \\ y_j + \phi &= f\left(k_j, l_j\right). \end{aligned}$$

Rearrange the first two conditions:

$$\begin{aligned} \left(\frac{w}{1-\alpha}\right)^{1-\alpha} &= s^{1-\alpha}\left(\frac{k_j}{l_j}\right)^{\alpha(1-\alpha)} \\ \left(\frac{r}{\alpha}\right)^{\alpha} &= s^{\alpha}\left(\frac{k_j}{l_j}\right)^{(\alpha-1)\alpha}, \end{aligned}$$

and multiply:

$$s = \left(\frac{w}{1-\alpha}\right)^{1-\alpha}\left(\frac{r}{\alpha}\right)^{\alpha}.$$

The profit maximization problem of an intermediate good firm is (apart from a constant having to do with fixed costs):

$$\max_{p_j,y_j} p_j y_j - s y_j,$$

After substituting out the demand curve:

$$y - s\frac{y}{p_j}.$$

According to this demand curve, the value of quantity demanded is always a constant. As a result, with higher prices revenues are constant, but of course costs a lower because quantity sold is less. There is no solution to the monopoly problem because for whatever p_j the monopolist contemplates, a higher price always brings in more profit.

If everyone has the same technology, then if a monopolist attempted to make positive profits, no matter how small, an entrant would come in and charge a slightly lower price to take all the business away from the monopolist. Thus, the monopolist who actually produces must make zero profits. Zero profits by the monopolist implies:

$$p_j y_j - s\left(y_j + \phi\right) = 0,$$

or, substituting in the production function, $f_j - \phi = y_j$:

$$p_j y_j = s f(k_j, l_j).$$

The firm markup is $\mu_j = p_j/s$. Dividing by p_j and taking the latter into account:

$$y_j = \frac{1}{\mu_j} f(k_j, l_j),$$

as requested.

Total costs break down into a part, $s\phi$, associated with the fixed cost and a part, sy_j, that is associated with the scale of operation. If the firm set $p_j = s$, then its revenues would match the part of its costs not related to fixed costs. To make zero profits, the firm must set price higher than s, so that revenues are enough to cover all its costs. This is why $\mu_j > 1$. If the scale of production is high, then the markup will be low because the fixed cost is relatively small. One can see this, by substituting out for y_j in terms of f in the above expression:

$$f(k_j, l_j) - \phi = \frac{1}{\mu_j} f(k_j, l_j),$$

or, after rearranging,

$$\mu_j = \frac{f(k_j, l_j)}{f(k_j, l_j) - \phi} = \frac{1}{1 - \phi/f(k_j, l_j)}.$$

Note that if f is high, then μ_j is low.

Since each firm's problem is symmetric, it will set the same price and hence it will have the same markup, μ, output, y, and inputs, k and l. Substituting this into the final good production function:

$$Y = \exp \int_0^1 \left[\log y_j\right] dj = \exp\left[\log y\right] = y = \frac{1}{\mu} f(k,l). \tag{2}$$

The share of income going to capital and labor may be computed from the efficiency conditions associated with cost minimization:

$$\begin{aligned} w &= s\,(1-\alpha)\left(\frac{k}{l}\right)^{\alpha} \\ r &= s\alpha\left(\frac{k}{l}\right)^{\alpha-1}. \end{aligned}$$

Thus,

$$\begin{aligned} wl + rk &= s\,(1-\alpha)\left(\frac{k}{l}\right)^{\alpha} l + s\alpha\left(\frac{k}{l}\right)^{\alpha-1} k \\ &= s\left[(1-\alpha)+\alpha\right] k^{\alpha} l^{1-\alpha} \\ &= sf(k,l) \\ &= s\mu Y, \end{aligned}$$

by (2). Now, $\mu = p/s$, where p denotes the price of the intermediate good producer. According to (1), $p = 1$ so that we can conclude

$$wl + rk = Y,$$

and income going to capital and labor is precisely equal to total final output. Intuitively, this is no surprise since there are zero profits.

Consider labor's share,

$$wl = s\,(1-\alpha)\left(\frac{k}{l}\right)^{\alpha} l = s\,(1-\alpha)\,f\,(k,l) = s\mu\,(1-\alpha)\,Y = (1-\alpha)\,Y.$$

Similarly, capital's share is $rk = \alpha Y$.

Conventional measures of TFP take total output and divide by capital, k, and labor, l, each raised to a power that corresponds to its share of income. In this case, that's just $f(k,l)$. So, TFP is $1/\mu$. If output responds to shocks outside the firm sector, then μ will fall when output is high and rise when output is low, i.e., it will be countercyclical. But, this means that estimated TFP is procyclical. An econometrician might be tempted to conclude that RBC theory is vindicated, in think ing that he/she has uncovered the shock that drives the business cycle. In this case, that would be a mistake because TFP is just responding endogenously to other things, and is not itself causal in this example.

3 A sequence of market equilibrium is a sequence of quantities, $\left\{c_t^t, c_{t+1}^t, k_{t+1}\right\}_{t=0}^{\infty}, c_0^{-1}$, and prices, $\{r_t, w_t, r_{k,t}\}_{t=0}^{\infty}$, such that each period's household and firm problems are satisfied and labor and capital markets clear.

The period t household problem is

$$\max_{c_t^t, c_{t+1}^t, k_{t+1}} \log(c_t^t)+\beta\log(c_{t+1}^t)+\lambda_{1t}\left[w_t - c_t^t - k_{t+1}\right]+\lambda_{2t}\left[r_{t+1}k_{t+1} - c_{t+1}^t\right].$$

The first order conditions are:

$$\begin{aligned} \frac{1}{c_t^t} &= \lambda_{1t} \\ \beta\frac{1}{c_{t+1}^t} &= \lambda_{2t} \\ -\lambda_{1t}+\lambda_{2t}r_{t+1} &= 0. \end{aligned}$$

Combining these, we obtain:

$$\frac{1}{c_t^t} = \beta\frac{1}{c_{t+1}^t}r_{t+1},$$

or, in the steady state equilibrium we consider:

$$\frac{c^o}{c^y} = \beta r. \tag{3}$$

$$\begin{aligned} w &= c^y + k \\ rk &= c^o \end{aligned} \tag{4}$$

The efficiency conditions of the firms imply:

$$
\begin{aligned}
r &= \alpha k^{\alpha-1} l^{1-\alpha} && (5)\\
w &= (1-\alpha)\, k^{\alpha} l^{-\alpha}. && (6)
\end{aligned}
$$

The resource constraint implies

$$c^y + c^0 + k = k^{\alpha} l^{1-\alpha}. \tag{7}$$

There are 7 equations, (3)-(7), $l = 1$ and 6 unknowns: r, w, c^y, c^o, k and l. There is one redundancy in these equations because the household budget constraints add up to the resource constraint (Walras' law) after imposing (5) and (6). To see this,

$$
\begin{aligned}
c^y + k &= \frac{(1-\alpha)\, k^{\alpha} l^{1-\alpha}}{l} \times l\\
c^o &= \alpha k^{\alpha} l^{1-\alpha},
\end{aligned}
$$

so that

$$c^y + k + c^o = (1-\alpha)\, k^{\alpha} l^{1-\alpha} + \alpha k^{\alpha} l^{1-\alpha} = k^{\alpha} l^{1-\alpha}.$$

Thus, we may drop one of the set of three equations: the three budget constraints and the resource constraint. In addition, from here on we impose $l = 1$. We drop the resource constraint. We can substitute out capital from the two household budget constraints, to obtain a single lifetime budget constraint:

$$w = c^y + \frac{c^o}{r}.$$

Imposing the intertemporal euler equation,

$$w = c^y + \frac{\beta r c^y}{r} \rightarrow w = c^y (1+\beta),$$

so that

$$
\begin{aligned}
c^y &= \frac{w}{1+\beta}\\
c^o &= \beta r \frac{w}{1+\beta}\\
k &= \beta \frac{w}{1+\beta}.
\end{aligned}
$$

These are just a rewrite of the households three equations. The equilibrium supplies two additional equations, (5) and (6):

$$\begin{aligned} r &= \alpha k^{\alpha-1} && (8) \\ w &= (1-\alpha)\, k^{\alpha}. && (9) \end{aligned}$$

Using this to substitute out for the wage in the household's capital decision:

$$k = \beta \frac{(1-\alpha)\, k^{\alpha}}{1+\beta},$$

which we can solve for capital:

$$k = \left[\frac{(1-\alpha)\,\beta}{1+\beta}\right]^{\frac{1}{1-\alpha}}.$$

This allows us to compute the return on capital:

$$r = \alpha k^{\alpha-1} = \frac{\alpha}{1-\alpha}\frac{1+\beta}{\beta},$$

as required. Note that the return on capital is zero when $\alpha = 0$. This is because capital is worthless in this case. Also, if β is zero, the return on capital is infinite because in this case, capital won't be accumulated and its marginal product will be infinite.

The object, r, is the rate of return on capital because a rate of return is the ratio of the total payoff on that asset to its price. The total payoff on capital in this model is just its rental rate because it completely depreciates in one period. The price of capital in this model is pinned down at unity by the technology.

For any parameters in which

$$\frac{1-\alpha}{\alpha} > \frac{1+\beta}{\beta},$$

we will have $r < 1$.

Note from (3) that when $r < 1$ consumption of the old is relatively low. Note from (7) that it is feasible to reallocate consumption from c^y to c^o if it is done one-for-one. That is, suppose we increase consumption of the old by Δ and reduce consumption of the young by the same amount. Consider what such a reallocation does to utility:

$$f(\Delta) = u(c^y - \Delta, c^o + \Delta) = \log(c^y - \Delta) + \beta \log\left(c^0 + \Delta\right)$$

We now ask what the slope of f is with respect to Δ, when the slope is evaluated at the equilibrium allocations. Differentiating,

$$f'(\Delta) = -\frac{1}{c^y - \Delta} + \frac{\beta}{c^0 + \Delta}$$

At $\Delta = 0$ and $\beta r = c^o / c^y$:

$$f'(0) = \frac{1}{c^o}\left[-\frac{c^o}{c^y} + \beta\right] = \frac{1}{c^o}[-\beta r + \beta] = \frac{\beta}{c^o}[1 - r] > 0$$

when $r < 1$. Thus, the reallocation increases the utility of each generation. This is because the terms of the intergenerational transfer (one-for-one) are better than those offered by the market. Note that this transfer increases the utility of each agent born in period 0, 1, 2, ..., as well as the utility of the current old. Their utility obviously rises because they simply receive a transfer without making any payment.

If there were a last date for the economy, then the last generation of initial young (who do not survive into old age) would be worse off under the transfer and so the wealth transfer is not obviously welfare improving. Of course, it might be.