

Homework #3
 Economics 411, Fall 2009
 Due Wednesday, October 14.
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1. (Boldrin-Montrucchio 1986). Consider the policy rule, $g : [0, 1] \rightarrow [0, 1]$:

$$g(x) = 4x(1 - x).$$

Draw this function, along with the 45 degree line, by hand in the unit box. Find an economy, (F, Γ, β, X) , for which the above function is the policy rule, where the economy satisfies all of our assumptions (i.e., assumptions A4.3-A4.9 in Stokey and Lucas). Here, x is the aggregate stock of capital at the beginning of the period and x' is its value at the end of the period.

Some hints: Recall, $\Gamma, \beta \in (0, 1)$ and F satisfy

$$g(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta v(y),$$

where

$$v(x) = F(x, g(x)) + \beta v(g(x)).$$

Recall that the definition of a function or a correspondence must include a specification of the domain and range. Recall too that a function, say $f(x, y)$, is strictly concave iff:

$$f_{xx}(x, y) < 0, \quad f_{yy}(x, y) < 0, \quad f_{xx}(x, y)f_{yy}(x, y) - f_{xy}(x, y)^2 > 0,$$

for all (x, y) in the domain of f . Also, it is easy to verify that

$$g(x) = \arg \max_{y \in \Gamma(x)} \Psi(x, y),$$

when $\Psi(x, y)$ is defined as follows:

$$\Psi(x, y) = -\frac{1}{2}y^2 + yg(x) - \frac{1}{2}Lx^2 + ax,$$

and L, a are known constants. Finally, note that $v(x) = \Psi(x, g(x))$, and use this to back out F . You can think of your task as having to identify values of a and L that ensure assumptions A4.3-A4.9 are satisfied. How many such values are there?

2. Suppose a planner chooses to maximize, by choice of $c_0, c_1, c_2, \dots$, the following expression:

$$u(c_0) + \delta[\beta u(c_1) + \beta^2 u(c_2) + \dots], u(c_t) = \log(c_t) \quad (1)$$

subject to

$$c_t = k_t^\alpha - k_{t+1}, 0 < \alpha < 1, c_t, k_{t+1} \geq 0, k_0 \text{ given,}$$

where $0 < \delta < \beta < 1$. When $\delta = 1$, this is the problem studied in exercises 2.2 and 4.9 in SL.

- (a) Let $k_{t+1} = g_t(k_t)$ denote the policy rule that solves this problem, $t = 0, 1, \dots$. From the perspective of period 0, the part of the problem from $t = 1$ and on looks exactly like the problem with $\delta = 1$. As a result, you know that the optimized value of $u(c_1) + \beta^2 u(c_2) + \dots$ has the form, $v(k_1)$, and you know how to compute $v(k_1)$ because it has a simple log-linear form. Use this to show that the optimal choice of k_1 has the form:

$$k_1 = g k_0^\alpha,$$

where g is a scalar. Derive an explicit formula relating g to the parameters of the model, β, α, δ . How does the saving rate from period $t = 1$ and on compare with the date 0 saving rate?

- (b) Is there a unique k^* with the property $k_t \rightarrow k^*$ as $t \rightarrow \infty$ for all k_0 ? Display a formula relating k^* to the parameters of the model.
- (c) Suppose $\beta = 1/1.03$, $\alpha = 1/3$, $\delta = 0.85$. Suppose $k_0 = k^*$. Display the values of $k_0, k_1, k_2, k_3, k_4, k_5$ that solve the problem as of date zero.
- (d) Now suppose that when date 1 occurs, the planner decides to reoptimize with respect to $k_2, k_3, \dots$. The initial condition for this problem is k_1 , the decision implemented by the planner last period. From the perspective of $t = 1$, the planner's preferences over c_t , $t \geq 1$ are as follows:

$$u(c_1) + \delta[\beta u(c_2) + \beta^2 u(c_3) + \dots]$$

and the resource constraint is as before. (Note how different the problem for $t \geq 1$ looks from the point of view of period 1 than it does from the point of view of period 0.) What values will the planner choose for k_1, k_2, k_3, k_4, k_5 ? If the planner chooses to re-optimize in this way every period, to what value will k_t actually tend?

- (e) Why are the values for k_2, k_3, k_4, k_5 chosen by the planner in date 1 different from the values planned for these variables as of date 0? Because of this difference, the problem is said to be *time inconsistent*. If δ had been set to one, we would not have had this problem. Why not?

Basically, the attitude of the planner is ‘I’m very impatient today (the discount rate from period 0 to period 1 is $\beta\delta$), but I’ll be less impatient tomorrow (the discount rate from period 1 to period 2 is β), so I’ll consume a lot today and save a lot tomorrow.’ Such an attitude is not time consistent because when tomorrow rolls around the planner says the same thing. In the end, the planner just ends up with a low capital stock. This type of model has been used to explain the behavior of smokers, who resolve that ‘tomorrow I’ll quit smoking, but tonight I’ll just have one or two more’. It also has been used to explain the low US saving rate. The notion is that many people say, ‘today I’ll spend, and tomorrow I’ll save’, day after day. (See the papers of David Laibson, of Harvard.) Does the solution that we have used in (d) make any sense? Would a rational person really make decisions in the time-inconsistent way described there?

3. Consider an economy with a large number of identical households, each having preferences, $\sum_{t=0}^{\infty} \beta^t u(c_t)$. Suppose the resource constraint is $c_t + i_t \leq f(k_t)$, where $k_{t+1} = i_t + (1 - \delta)k_t$, f is strictly increasing and concave, $f'(k) \rightarrow \infty$ as $k \rightarrow 0$, $f'(k) \rightarrow 0$ as $k \rightarrow \infty$, $0 < \delta < 1$. Assume investment, i_t , is irreversible, i.e., it must be that $i_t \geq 0$. In addition, suppose $c_t, k_t \geq 0$ and that $k_0 > 0$ is given. Consider the functional equation associated with this problem:

$$\begin{aligned} v(k) &= \max_{k' \in \Gamma(k)} u(f(k) + (1 - \delta)k - k') + \beta v(k') \\ \Gamma(k) &= \{k' : (1 - \delta)k \leq k' \leq f(k)\}. \end{aligned}$$

- (a) State a set of assumptions on β and u that guarantee there is a unique, differentiable, concave v that solves the above functional equation. For each property of v , explain which assumptions are used to get it.
 - (b) Show that monotonicity of $\Gamma(k)$, Assumption 4.6 in S-L, fails so that one of the conditions of Theorem 4.7 which guarantee strictly increasing v , is not satisfied.
 - (c) Show that the feasible set for this economy satisfies the following ‘quasi-monotonicity property’: if $\tilde{k} \geq k$, then $\Gamma(k) + (1 - \delta)(\tilde{k} - k) \subseteq \Gamma(\tilde{k})$. Here, the sum of a set, say X , and a number, say a , is a new set, $X + a$, where $X + a \equiv \{x + a : x \in X\}$.
 - (d) Show: v is an increasing function in k . (Hint: (i) following the basic strategy of the proof of Theorem 4.7, it’s enough to establish that the assumptions of Theorem 4.7 with the monotonicity assumption on Γ replaced by quasi-monotonicity guarantee Tw is increasing if w is; (ii) make use of the fact that if $k' \in \Gamma(k)$, then $\tilde{k}' = k' + (1 - \delta)(\tilde{k} - k) \in \Gamma(\tilde{k})$, $\tilde{k}' > k'$, and $f(\tilde{k}) + (1 - \delta)\tilde{k} - \tilde{k}' > f(k) + (1 - \delta)k - k'$.) Can you provide intuition for the fact that v is increasing even though Γ fails to satisfy monotonicity?
4. Do exercise 6.7a-f, pages 157-158 in S-L. Ex. 6.7d asks you to verify the existence of a two period cycle. For extra credit, do exercise 6.7g, which explores the ‘stability’ of that cycle.