Homework #5
Economics 411
Due Wednesday, October 28
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1. Consider the following functional equation:

$$T(v) = \max_{0 \leq \lambda \leq A+1-\delta} \frac{[A+1-\delta-\lambda]^{(1-\sigma)}}{1-\sigma} + \beta\lambda^{(1-\sigma)}v.$$

Suppose $\sigma > 1$, $A > \delta$, and $\beta(A+1-\delta)^{1-\sigma} < 1$. Note that the 'function', v, on which T operates is actually a point.

(a) Show: $T(v) = \infty$ for $v > 0$, $T(0) = \frac{[A+1-\delta]^{(1-\sigma)}}{1-\sigma}$.

(b) Show: the derivative of T at $v = v_0 < 0$ is:

$$\frac{dT(v_0)}{dv} = \beta\lambda(v_0)^{(1-\sigma)},$$

where

$$\lambda(v_0) = argmax_{0 \leq \lambda \leq A+1-\delta} \frac{[A+1-\delta-\lambda]^{(1-\sigma)}}{1-\sigma} + \beta\lambda^{(1-\sigma)}v_0.$$

(c) Show that T does not satisfy the conditions of Blackwell's theorem.

(d) What happens to $\lambda(v)$ as $v \to -\infty$?

(e) What does the graph of $T(v)$ versus v for $v \leq 0$ look like? Does it cross a 45^0 line drawn in the negative orthant? Draw the $T(v)$ function by hand, emphasizing its qualitative features.

(f) Explain, using the graph you just developed, why $T^j v_0 = v^*$ as $j \to \infty$, for every $v_0 < 0$. Also, explain why v^* such that $v^* = T(v^*)$, is unique. Explain how T has many of the characteristics of Theorem 4.16, without satisfying all the assumptions.

2. Consider the neoclassical growth model in homework 3, in which $\delta \neq \beta$. Let a sequence of markets equilibrium be a set of sequences,

$$\{r_t, w_t, c_t, y_{t,}, \ k_{t+1}, n_t; t = 0, 1, ..., \infty\},$$

such that (i) for each t, c_t, k_{t+1}, n_t are the period t part of the solution to the household problem, (ii) for each t, y_t, n_t, k_t solve the representative firm problem, and (iii) $c_t + k_{t+1} - (1 - \delta) k_t \leq k_t^\alpha n_t^{1-\alpha}$ for each t.

   (a) Explain how this definition of a sequence of markets equilibrium is incomplete. Fix the statement of the household problem so that that problem is well-specified.

   (b) With the cleaned up definition of a sequence of markets equilibrium, does there exist such an equilibrium? Explain carefully.

   (c) Describe the quantities in the Arrow-Debreu equilibrium and explain why that the complications associated with the sequence of market equilibrium are not present in the Arrow-Debreu equilibrium.

3. Consider again the economy in homework #3, only now consider the efficient allocation problem associated with that economy. We call this allocation problem a 'planning' problem. You can also think of the planning problem as the problem of an agent ('Robinson Crusoe') who has the utility function specified, and who has the technology specified. Imagine that the planner cannot commit to his future actions. That is, in making his decision at each date the planner ignores any past commitments that might have been made in the past and he assumes that he will do the same in the future. A way to think of what will happen in such a situation is to pretend that the planner at each date is actually a different person. Thus, when the planner makes a decision at any particular date, he takes the policy rule of all future planners as given (past policy rules are irrelevant, once the capital stock at the start of the period is known). The solution concept that seems reasonable here is a policy rule for all planners that satisfies a particular fixed point. That is, the policy rule has the property that conditional on all future planners following it, policy rule is optimal for the current planner. This solution resembles a Nash equilibrium. Specifically, suppose

a planner optimizes today, subject to the constraint that planners at all future dates save according to the rule, $k_{t+1} = dk_t^\alpha$. Let $v(k_t; d)$ denote the present value of utility, $u(c_t) + \beta u(c_{t+1}) + \ldots$, that occurs when the saving rate, d, is followed starting in period t, and going forever into the future.

(a) Display an explicit formula for $v(k_t; d)$ (hint: you can find it as the fixed point of a dynamic programming sequence that you can do with paper and pencil, like we did in class)

(b) Let $g(k; d)$ denote the policy rule of a planner who expects the saving rate in the future to be d. Show that $g(k; d) = D(d)k^\alpha$, and derive an explicit formula for $D(d)$.

(c) Let d^* denote the fixed point mentioned above, so that $d^* = D(d^*)$. Display a formula relating d^* to the parameters of the model.

(d) Call the decisions, $k_0, k_1, \ldots$ emerging from the solution in (c) the *Nash solution.* Call the decisions that are optimal from the perspective of the first date, the *commitment solution.* Compare the value of utility in these solutions, assuming $k_0 = 1$ in each case.

(e) Now consider a large economy version of the above setup, in which there are many identical households and a representative firm. Which of the allocations in (d) corresponds to the allocations in a sequence of markets equilibrium, and which corresponds to the allocations in an Arrow-Debreu equilibrium? Explain.