

Homework #7

Economics 411

Fall, 2009, Due Wednesday, November 18.

Christiano

1. Entrepreneurs have access to a technology for converting capital, k , into output

$$\omega k^\alpha,$$

where ω is a technology shock drawn independently by each entrepreneur from a distribution with $E\omega = 1$ and cumulative distribution function $F(x) \equiv \text{prob}[\omega \leq x]$. The realization of ω is observed by the entrepreneur, and can be seen by a lender only if the lender pays a monitoring cost, μk^α . The depreciation rate on capital is δ , so that after production, entrepreneurs have $(1 - \delta)k$ units of capital left, which they can sell at a price of unity (the price of output is the numeraire). Thus, after production the entrepreneur who draws ω has the following resources:

$$y(\omega) = \omega k^\alpha + (1 - \delta)k, \quad 0 < \delta, \alpha \leq 1,$$

At the beginning of the period, entrepreneurs have no capital, but they do have net worth, n , that they can use for purchasing k . Suppose there are many entrepreneurs with each possible level of n .

Consider an entrepreneur with net worth, n , who purchases an amount of capital, $k > n$. The entrepreneur borrows $b \equiv k - n$ at gross rate of interest, Z , from a bank. There is a large number of banks that specialize in lending to entrepreneurs with each level of net worth, n , and there is free entry into banking. In case the entrepreneur's revenue, $y(\omega)$, falls below the required payment to the bank, $Z(k - n)$, the entrepreneur declares bankruptcy and is monitored. In addition, the bank takes whatever the entrepreneur has. Let $\bar{\omega}$ be defined by

$$\bar{\omega} k^\alpha + (1 - \delta)k = Z(k - n).$$

Prior to making loans to entrepreneurs, banks have access to a competitive market in which they can borrow as much or as little as they want, at gross rate of interest, R . At the end of the period, when the banks have to repay household loans, the only source of funds available

to them is the funds given to them by entrepreneurs. Entrepreneurial utility prior to production is proportional to their expected end-of-period resources, net of bank costs.

- (a) Show that the expected profits of an entrepreneur with net worth n , interest rate Z , and loan amount $k - n$ can be written

$$[1 - \Gamma(\bar{\omega})] k^\alpha,$$

where

$$\Gamma(\bar{\omega}) = [1 - F(\bar{\omega})] \bar{\omega} + \int_0^{\bar{\omega}} \omega dF(\omega).$$

- (b) Suppose that each bank deals with a large, randomly selected set of entrepreneurs. Show that the average revenues, across all loans to entrepreneurs with net worth n , is

$$[\Gamma(\bar{\omega}) - \mu F(\bar{\omega})] k^\alpha + (1 - \delta) k.$$

- (c) Display the zero profit condition for the banks that lend to entrepreneurs with net worth, n . Write this in terms of the variables, b , R , Z and n only (not $\bar{\omega}$ or k). Can there be an equilibrium in which banks offer an interest rate Z and banks allow entrepreneurs to borrow as much as they want at that interest rate? Explain.
- (d) Let the bank zero profit condition define a menu of contracts, (b, Z) , that banks who lend to entrepreneurs with net worth n offer in equilibrium. Display a constrained optimization problem that characterizes which b , Z combination entrepreneurs with net worth, n , select from this menu. You may assume that the chosen contract is interior and is uniquely characterized by the first order conditions evaluated at equality.
- i. Will the interest rate in the selected contract vary with entrepreneurial net worth, n ? Does your answer to the last question change across the cases, $\alpha = 1$ and $\alpha < 1$? Establish your answer carefully.
 - ii. Define leverage, L , as

$$L = \frac{b + n}{n}.$$

Show that L is independent of n when $\alpha = 1$.

2. (Vintage capital interpretation of exogenous, embodied technical change model.) Consider an economy with capital of different vintages. At time t , the amount of capital of vintage τ , $k_{t,\tau}$, $\tau = 1, 2, 3, \dots$, is

$$k_{t,\tau} = \gamma^{t-\tau}(1 - \delta)^{\tau-1}i_{t-\tau},$$

where $\gamma > 1$, $0 < \delta < 1$, $i_{t-\tau}$ is the amount of investment, in time $t - \tau$ consumption units, applied in period $t - \tau$. Capital which has vintage τ in period t has vintage $\tau + 1$ in period $t + 1$. Investment expenditures at time t , i_t , must all be applied to the latest vintage (for a model in which investment in old vintages is feasible and desirable, see Chari and Hopenhayn, JPE, 1991) and results in $k_{t+1,1} = \gamma^t i_t$ units of new-vintage period $t + 1$ installed capital goods. Consider a given amount of investment, i . Note that this investment applied in period $t + 1$ produces more new-vintage installed capital (i.e., $\gamma^{t+1}i$) than the same level of investment applied in period t (i.e., $\gamma^t i$). This reflects the assumption, $\gamma > 1$ which is designed to capture the notion that there is exogenous technical progress that is embodied in new capital equipment. Note that the efficiency of a particular vintage stays constant over time, it's just that the efficiency of each succeeding vintage is greater than the efficiency of the previous one.

Capital of each vintage is operated with labor to produce a homogeneous output good, $y_{t,\tau}$ according to the following production function:

$$y_{t,\tau} = k_{t,\tau}^\alpha n_{t,\tau}^{1-\alpha}, \quad 0 < \alpha < 1, \quad \tau = 1, 2, 3, \dots$$

Suppose there is a competitive market in capital of different vintages and in labor. Each vintage of capital has the same rental rate, r_t , since capital is measured in common efficiency units. Similarly, the wage rate is w_t .

- (a) Show that a firm's profit maximizing choice of $n_{t,\tau}$ gives rise to the following relationships:

$$y_t = k_t^\alpha n_t^{1-\alpha}, \quad (1 - \alpha) \left(\frac{k_t}{n_t} \right)^\alpha = w_t, \quad \alpha \left(\frac{k_t}{n_t} \right)^{\alpha-1} = r_t,$$

where

$$y_t = \sum_{\tau=1}^{\infty} y_{t,\tau}, \quad k_t = \sum_{\tau=1}^{\infty} k_{t,\tau}, \quad n_t = \sum_{\tau=1}^{\infty} n_{t,\tau}.$$

(b) Show that ‘aggregate capital’, k_t , evolves as in the Solow II model:

$$k_{t+1} = (1 - \delta)k_t + \gamma^t i_t.$$