Christiano
Econ 411, Fall 2009

MIDTERM EXAM

There are four questions, each having an equal number of points. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have 2 hours. Good luck!

1. Let $(\beta, \Gamma, F, X)$ satisfy assumptions 4.3-4.9 in S-L. For this question, you may refer to any theorem in S-L without proving it. Consider the problem

$$\begin{aligned} & \max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \qquad (1) \\ \text{subject to} \quad & x_{t+1} \in \Gamma(x_t), \text{ for all } t, \\ & x_0 > 0 \text{ given.} \end{aligned}$$

Additionally, assume that $0 \in \Gamma(x)$, for all $x \in X$. Let $\{x_t^*\}_{t=0}^{\infty}$ be a solution to the problem in (1). Show that

(a) $F_2(x_t^*, x_{t+1}^*) + \beta F_1(x_{t+1}^*, x_{t+2}^*) = 0$, for all $t$.

(b) $\lim_{t \to \infty} \beta^t F_1(x_t^*, x_{t+1}^*) x_t^* = 0$.

2. Consider an economy, in which all households are identical and have preferences

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - l_t).$$

the resource constraint in per capita terms is

$$c_t + k_{t+1} - (1 - \delta) k_t \leq f(k_t, l_t),$$

where $c_t, k_{t+1} \geq 0$ denote consumption and capital, respectively, and $0 \leq l_t \leq 1$ denotes hours worked. Utility is strictly increasing, strictly concave, and $f$ is homogeneous of degree one, strictly increasing. As you answer this question, make up whatever additional assumptions about $u$, and $f$ you feel you need.

(a) Define an Arrow-Debreu date 0 equilibrium for this economy, by giving households a budget constraint and firms the production technology. Set up markets and prices.

(b) Establish the following properties of the Arrow-Debreu equilibrium: (i) if a system of prices and profits are part of an equilibrium, then any positive scalar multiple of these prices and profits are also an equilibrium; (ii) all prices are strictly positive; (iii) firm profits are zero; (iv) a particular relationship must hold between the return on capital and the wage rate.

(c) Define the efficient allocations as the allocations which maximize the representative household's utility subject to the resource constraint. Prove that the allocations in an Arrow-Debreu equilibrium are efficient.

(d) Set up a sequence of markets equilibrium. Show that the allocations in a sequence of markets equilibrium and in an Arrow-Debreu equilibrium satisfy the first order necessary conditions associated with the social planner's problem.

3. Consider an Arrow-Debreu economy with $n$ firms. All firms use the same linear homogeneous technology, $y_t^i = f(k_t^i, l_t^i)$, where $f$ is strictly increasing, strictly concave in its first and second arguments, and continuously differentiable. Firms buy production inputs in competitive markets and take the rental rate of capital and the wage rate as given. Let

$$k_t \equiv \frac{1}{n}\sum_{i=1}^{n} k_t^i,\ l_t \equiv \frac{1}{n}\sum_{i=1}^{n} l_t^i,\ y_t \equiv \frac{1}{n}\sum_{i=1}^{n} y_t^i.$$

(a) Show that profit maximizing input choices imply that $k_t^i/l_t^i$ is the same for all $i = 1, \dots, n$, for all $t$.

(b) Show that

$$y_t = f(k_t, l_t)$$

has to hold in every Arrow-Debreu equilibrium, for all $t$.

(c) When an Arrow Debreu equilibrium is defined, we typically assume that there are many firms and they all make identical choices (this is the 'representative firm' assumption). For example, the objects in the A-D equilibrium typically only refer to economy-wide aggregates (i.e., for this economy, we include things like $k_t$, not $k_t^i$, for all $i$). Consider an alternative definition of an A-D equilibrium, the A-D$^a$ equilibrium, in which the equilibrium objects also include a specification of actions at the level of individual firm. Define carefully this equilibrium concept. Show that corresponding to each A-D equilibrium, there is a large range of A-D$^a$ equilibria, where the objects without firm index coincide across the A-D and A-D$^a$ equilibria, but the objects with firm index are different.

4. Consider the following neoclassical economy. Household preferences are:

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \; 0 < \beta < 1,$$

and technology is given by

$$c_t + k_{t+1} \leq f(k_t),$$

where

$$f' \underset{k \to 0}{\to} \infty, \; f(0) = 0, \; f' \underset{k \to \infty}{\to} 1 - \delta, \; u' \underset{c \to \infty}{\to} 0, \; u' \underset{c \to 0}{\to} \infty, \; u' > 0 \text{ for } c > 0,$$

$0 < \delta < 1$ and $f, u$ are twice continuously differentiable and strictly concave. Also, $k \in K \equiv \left[0, \bar{k}\right]$, where $\bar{k}$ the maximal feasible capital stock. The efficient allocation problem is to find sequences, $\{c_t, k_{t+1}\}_{t=0}^{\infty}$, that maximize utility subject to the technology constraint, the given $k_0$ and $c_t, k_{t+1} \geq 0$.

(a) What is the smallest number that can serve as $\bar{k}$? Explain carefully.

(b) Explain in intuitive terms why this problem can be solved by first finding the (unique) functions, $v$ and $g$, with the property

$$v(k) = \max_{0 \leq k' \leq f(k)} u(f(k) - k') + \beta v(k')$$

(hint: write out the efficient allocation problem and exploit its structure).

(c) Sketch a proof that $v$ is strictly concave. Make clear what about this proof is hard and subtle, and what part of this proof is straightforward.

(d) Prove that there exists a unique $k^* > 0$ that satisfies $k^* = g(k^*)$, and display a simple formula for $k^*$ that does not involve having to know what the $g$ function is.