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D11-2, Winter 1996

FINAL EXAM ANSWERS

1. Answer for question 1.

   i. Sequence of markets equilibrium. At each date, t, the household maximizes discounted utility from then on:

$$\sum_{j=t}^{\infty} \beta^{j-t} u(c_j),$$

subject to a sequence of budget constraints:

$$c_j + i_{pj} \leq r_j k_{pj} + w_j n - T_j,\; j \geq t,$$

where w_j and r_j are market prices beyond the control of the household. The household uses its entire endowment of time for labor effort, n, because it does not value leisure. The firms choose n_t and $k_{p,t}$ such that profits are maximized, where profits are defined as follows:

$$k_{g,t}^{\gamma} n_t^{(1-\alpha)} k_{pt}^{\alpha} - w_t n_t - r_t k_t.$$

A sequence of markets equilibrium is a set of prices and quantities, $\{r_t, w_t; t \geq 0\}$, $\{y_t, c_t, n, i_{pt}, i_{gt}; t \geq 0\}$ and taxes, $\{T_t; t \geq 0\}$ such that

- given taxes and prices, the quantities solve the household problem.
- given the prices, the quantities solve the firm problem.
- given the quantities and a value of s, the government budget constraint is satisfied.
- the resource constraint is satisfied.

(a) the first order condition for the household is

$$u_{c,t} = \beta u_{c,t+1}[r_{t+1} + 1 - \delta_p],$$

and the firm sets $f_{k_{p},t+1} = r_{t+1}$, where $f_{k_{p},t+1}$ is the marginal product of private capital. Combining these, and taking functional forms into account:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \beta\left[\alpha\left(\frac{nk_{g,t+1}}{k_{p,t+1}}\right)^{(1-\alpha)} + 1 - \delta_p\right].$$

Let g_c denote the gross growth rate of consumption in a balanced growth path. Then,

$$(g_c)^{\gamma} = \beta[\alpha(ns)^{(1-\alpha)} + 1 - \delta_p].$$

Suppose g_c corresponds to some given positive net growth rate, ie., $g_c > 1$. Then,

$$s = \frac{1}{n}\left\{\frac{1}{\alpha}\left[\frac{g_c^{\gamma}}{\beta} + \delta_p - 1\right]\right\}^{\frac{1}{1-\alpha}}.$$

The number in square brackets is positive, so that s is well defined. Thus the Euler equation is consistent with constant consumption growth in steady state. To fully answer the question, we need to establish (i) that the other equations - the household budget equation and the resource constraint - are also satisfied with a constant consumption growth rate and (ii) that the other quantity variables display positive growth too. Let g_g and g_p denote the gross growth rates of government and private capital, respectively. Then, the government's policy for choosing $k_{g,t}$ implies:

$$g_g = g_p = g,$$

say. Note that output can be written

$$k_{gt}^{(1-\alpha)}k_{pt}^{\alpha}n^{(1-\alpha)} = k_{gt}(k_{pt}/k_{gt})^{\alpha}n^{(1-\alpha)} = k_{gt}s^{\alpha}n^{(1-\alpha)}.$$

Divide the resource constraint by k_{gt}:

$$\frac{c_t}{k_{gt}} + g_{t+1} - (1-\delta_g) + g_{t+1} - (1-\delta_p) = s^{\alpha}n^{(1-\alpha)}.$$

So, in a constant growth steady state (i.e., $g_{t+1} = g$, constant) the consumption to public capital ratio is a constant, equal to the following:

$$s^{\alpha} n^{(1-\alpha)} + (1-\delta_g) + (1-\delta_p) + 2g.$$

But, the consumption to public capital ratio being constant implies:

$$g_c = g.$$

The household budget constraint is trivially satisfied, since it is equivalent with the resource constraint given the first order conditions of firms, linear homogeneity of the production function with respect to firms' choice variables, and the government budget constraint.

(b) The planner's problem is: choose $c_t, k_{g,t+1}, k_{p,t+1}$, $t \geq 0$ to maximize discounted utility. After substituting out consumption using the resource constraint, the problem becomes:

$$\max_{\{k_{g,t+1},k_{p,t+1}\}} \sum_{t=0}^{\infty} \beta^t u[k_{gt}^{(1-\alpha)} n^{(1-\alpha)} k_{pt}^{\alpha} + (1-\delta_g)k_{g,t} + (1-\delta_p)k_{p,t} - k_{p,t+1} - k_{g,t+1}],$$

subject to the object in square brackets (consumption) being nonnegative at all dates, and to $k_{g,t}, k_{p,t} \geq 0$. The planner's first order conditions are:

$$\begin{aligned} u_{c,t} &= \beta u_{c,t+1}[f_{k_{p,t+1}} + 1 - \delta_p] \\ u_{c,t} &= \beta u_{c,t+1}[f_{k_{g,t+1}} + 1 - \delta_g], \end{aligned}$$

for $t = 0, 1, 2, \ldots$. With the functional forms:

$$\begin{aligned} \left(\frac{c_{t+1}}{c_t}\right)^{\gamma} &= \beta\left[\alpha k_{g,t+1}^{\gamma} \left(\frac{n}{k_{p,t+1}}\right)^{(1-\alpha)} + 1 - \delta_p\right] \\ \left(\frac{c_{t+1}}{c_t}\right)^{\gamma} &= \beta[\gamma (k_{g,t+1})^{\gamma-1} n^{(1-\alpha)} (k_{p,t+1})^{\alpha} + 1 - \delta_g]. \end{aligned}$$

Substituting out consumption using the resource constraint, these two equations represent a vector difference equation in k, k', k'', where $k = [k_g\ k_p]'$. There are many solutions to this equation that are consistent with the given initial condition, $k_0 = [k_{g,0}\ k_{p,0}]$. One can construct the whole family of solutions by indexing them by k_1: different values of k_1 give rise, by iterating on the euler equation, to different sequences of capital. Not all are optimal. Only the one solution that also satisfies the transversality condition is optimal. Thus, satisfying the Euler equation is not sufficient for an optimum.

(c) Setting $\gamma = 1 - \alpha$ and equating the planner's two first order conditions, we get:

$$\begin{aligned}&\beta\left[\alpha\left(\frac{nk_{g,t+1}}{k_{p,t+1}}\right)^{(1-\alpha)} + 1 - \delta_p\right]\\ = \quad &\beta\left[(1-\alpha)n^{(1-\alpha)}\left(\frac{k_{p,t+1}}{k_{g,t+1}}\right)^{\alpha} + 1 - \delta_g\right],\end{aligned}$$

which requires that $\frac{k_{p,t+1}}{k_{g,t+1}}$ be a particular constant for $t = 0, 1, \ldots$. Call this constant s^*. By setting $s = s^*$ the government cannot do better, since this achieves the planner's optimum.

2. To apply the Benveniste and Scheinkman theorem, first establish that $h \in \mathrm{int}(n : \bar{n}(k', k') \le n \le 1)$. To do this, note first that $u[f(k', n) - k', n]$ is differentiable in n because (by P1 and T1) u and f are. In particular, this derivative is

$$r(n) = u_c[f(k', n) - k', n]f_n(k', n) + u_n[f(k', n) - k', n].$$

As $n \to 1$, $r(n) \to -\infty$ because $u_n \to -\infty$ (by P3) and because u_c and f_n go to well-defined finite numbers (see P7). Suppose $\bar{n} > 0$. As $n \to \bar{n}$, $r(n) \to \infty$ because $u_c \to \infty$ (by P2) and u_n, f_n go to finite numbers (see P6). Suppose $\bar{n} = 0$. As $n \to 0$, $r(n) \to \infty$ for the reasons just given, plus the fact that now $f_n \to \infty$ too (see T3). Since r is continuous, the fact that it passes from ∞ to $-\infty$ as n goes from $\bar{n}(k', k')$ to 1, implies that $r(n)$ is zero somewhere on the interior of this set. But, by P1 and T1, $u[f(k', n) - k', n]$ is strictly concave in n, so that a zero derivative is necessary and sufficient for an optimum. It follows that the value of n where $r(n) = 0$ is $h(k^0, k')$. This establishes $h(k^0, k') \in \mathrm{int}(n : \hat{h}(k^0, k') \leq n \leq 1)$.

Next, it is easily established that there is a neighborhood, D, of k^0 such that $k \in D$ implies $h(k^0, k')$ is feasible, i.e., $\hat{h}(k, k') \leq h(k^0, k') \leq 1$. This follows from our interiority result for h, and by the continuity of $\hat{h}(k, k')$ for $k \in D$. The latter reflects continuity of u and f (see P1 and T1).

Define

$$W(k) = u[f(k, h(k^0, k')) - k', h(k^0, k')].$$

By feasibility of $h(k^0, k')$ and the definition of $F(k, k')$, it follows that:

$$\begin{aligned} F(k, k') &\geq W(k),\ k \in D \\ F(k^0, k') &= W(k^0). \end{aligned}$$

Also, by P1 and T1, $W(k)$ is differentiable and concave for all $k \in D$. All we need to apply the Benveniste and Scheinkman theorem is concavity of $F(k, k')$. In class (and the handout) we got strict concavity, principally from strict concavity on u and f (P1 and T1). But, we also used that f is strictly increasing in k and n (T1) and that f_n is strictly increasing in k (T6).

To apply the Benveniste and Scheinkman theorem, let $V(x)$ and x be $F(k, k')$ and k, respectively (k' is treated as a constant throughout.) Let x_0 be k^0. It then follows that $F(k, k')$ is differentiable at $k = k^0$ and that the derivative there is the derivative of $W(k)$ at $k = k^0$. That derivative, trivially, is

$$u_c[f(k^0, h(k^0, k')) - k', h(k^0, k')] f_k(k^0, h(k^0, k')).$$

3. Question 3.

   (a) A sequence of markets competitive equilibrium is a set of prices,

   $$\{p(s^t), w(s^t), r(s^{t+1}), R(s^t),\ \text{all } t, s^t\}$$

   and quantities,

   $$\{Z(s^t), B(s^t), k(s^t), n(s^t), c(s^t), y(s^t),\ \text{all } t, s^t\}$$

   such that, for each t, s^t:

- given the prices, the quantities solve the household problem.
- given the prices, the quantities solve the firm problem, with free entry.
- the resource constraint is satisfied.

(b) The household fonc's for each t, s^t:

$$
\begin{aligned}
u_c(s^t) &= \beta \sum_{s^{t+1}|s^t} \mu(s^{t+1} \mid s^t) u_c(s^{t+1}) r(s^{t+1}) \\
u_c(s^t) &= \beta \sum_{s^{t+1}|s^t} \mu(s^{t+1} \mid s^t) u_c(s^{t+1}) R(s^t)
\end{aligned}
$$

To get the firm fonc's, first substitute out for capital from its finance constraint (with strict equality). Then, the fonc's for $n(s^{t+1}); Z(s^t); B(s^t)$ are, respectively:

$$
\begin{aligned}
f_n(s^{t+1}) &= w(s^{t+1}) \\
0 &= \sum_{s^{t+1}|s^t} \mu(s^{t+1} \mid s^t) u_c(s^{t+1}) [f_k(s^{t+1}) + 1 - \delta - r(s^{t+1})] \\
0 &= \sum_{s^{t+1}|s^t} \mu(s^{t+1} \mid s^t) u_c(s^{t+1}) [f_k(s^{t+1}) + 1 - \delta - R(s^t)];
\end{aligned}
$$

where f_n and f_k denote the partial derivatives of the production function with respect to n and k, respectively.

Combining the household and firm fonc's, we get:

$$
\begin{aligned}
0 &= u_n(s^t) + u_c(s^t) f_n(s^t) \\
u_c(s^t) &= \beta \sum_{s^{t+1}|s^t} \mu(s^{t+1} \mid s^t) u_c(s^{t+1}) [f_k(s^{t+1}) + 1 - \delta];
\end{aligned}
$$

for all t, s^t. It is easily verified that these first order conditions correspond to the solution of the planning problem.

(c) That $\pi(s^t) = 0$ follows from two facts: (i) ex ante profits for the firm are zero because of free entry, and (ii) ex ante profits are a weighted average of ex post profits one period hence. Because the weights are positive, and because ex post profits are non-negative by the cash flow constraint, it follows that profits must be zero in each state of nature and each time period. We can use this result to derive the required expression for $r(s^{t+1})$. Note that by linear homogeneity, output is $f_k(s^{t+1})k(s^t) + f_n(s^{t+1})n(s^{t+1})$. Using firm's first order condition for labor and substituting into the $\pi(s^{t+1}) = 0$ equation, we get

$$[f_k(s^{t+1}) + 1 - \delta](Z(s^t) + B(s^t)) - r(s^{t+1})Z(s^t) - R(s^t)B(s^t) = 0.$$

Divide by $Z(s^t)$ and impose the definition of $\gamma(s^t)$, and the result follows.

(d) For all $\gamma(s^t)$, the quantity allocations of the competitive equilibrium coincide with the allocations in the planner's problem. The latter are independent of the value of $\gamma(s^t)$.

(e) Rewriting the household fonc:

$$R(s^t) = \frac{1}{\sum_{s^{t+1}|s^t} \pi(s^{t+1} \mid s^t)m(s^{t+1})} = \frac{1}{E_t m_{t+1}},$$

where $m(s^{t+1}) = \beta u_c(s^{t+1})/u_c(s^t)$. In this notation, the household fonc for equity is:

$$1 = \sum_{s^{t+1}|s^t} \pi(s^{t+1} \mid s^t)m(s^{t+1})r(s^{t+1}) = E_t m_{t+1} r_{t+1},$$

where, hopefully, the simplified notation is not misleading. Applying the formula in the hint,

$$\begin{aligned} E_t m_{t+1} E_t r_{t+1} &= E_t m_{t+1} r_{t+1} - Cov_t(m_{t+1}, r_{t+1}) \\ &= 1 - Cov_t(m_{t+1}, r_{t+1}) \end{aligned}$$

But, $E_t m_{t+1} E_t r_{t+1} = E_t r_{t+1}/R_t$ is by definition the equity premium, $P(s^t)$. Substituting for r in the covariance formula, one gets

$$P_t = 1 - Cov_t(m_{t+1}, r_{t+1}) = 1 - (1 + \gamma_t)Cov_t(f_{k,t+1}, m_{t+1}),$$

where γ_t corresponds to $\gamma(s^t)$. The reason this substitution works the way it does is that γ_t and R_t are constants relative to the date t information set and because

$$Cov_t(f_{k,t+1} + a_t, m_{t+1}) = Cov_t(f_{k,t+1}, m_{t+1}),$$

for any a_t that is constant relative to the time t information set. The result sought follows from two observations. First, real allocations (hence, $Cov_t(f_{k,t+1}, m_{t+1})$) are independent of γ_t; and second, the conditional covariance is negative.

4. Question 4. See the answer to question 1 in homework #7.

5. Question 5.

   (a) The real business cycle story is that the cycle is driven by a shock that rotates all production functions up. Given no change in work effort, worker productivity would rise. But, there is a (small) increase in employment. The increase by itself drives labor productivity down because of concavity, but not by enough to compensate for the productivity effects of the positive rotation of the production function. Thus, labor goes up and productivity goes up. There is some disenchantment with this explanation. Primarily, this reflects that the measured `productivity' shock seems correlated with things (like monetary policy and military spending) that it should not be closely related to in the short run if it were really productivity.

   (b) According to the labor hoarding idea, when productivity goes up in an expansion, that primarily reflects a systematic measurement error. The rise in measured productivity in times like this simply reflects increased work effort by workers that is not picked up in the numbers, but which does not correspond to any technological innovation.

   (c) The increasing returns models say the `increasing returns' to labor may reflect nonconcave production functions, where productivity automatically increases with an increase in employment. There are two types of such models: one in which the increasing returns is `internal' to the firm, and the other in which the increasing returns is external to the firm.