1. (Human capital) Consider the endogenous growth problem in homework #4. The parameters The recursive representation of that problem is:

\[
    v(x) = \max_{x_0 \in \mathbb{R}^2} \{ f(x; x_0, y) - y^\beta v(x_0) g(x) \}
\]

where

\[
    j(x) = f(x_0, y; 1 - y - 1 + \epsilon \;); \quad (1) \quad 2
\]

problem of a planner who seeks to maximize

\[
    \chi^{\frac{1}{\gamma}} - t(c_t^{\frac{\beta}{\gamma}}); \quad 0 < \beta < 1;
\]

subject to the following restrictions. The resource constraint is:

\[
    c_t + k_t + \mu k_t = k_t^{(1 + \mu)};
\]

where \( k_t \) denotes beginning-of-period \( t \) physical capital, and \( n_t \) denotes labor, measured in efficiency units (as opposed to numbers of people, or time). At date 0, \( k_0 \) is given. Also, the following nonnegativity constraints must be respected:

\[
    c_t, 0; k_t, 0; \quad (1 \; \leq k_t; \; t = 0; 1; \cdots)
\]

The constraints on labor are:

\[
    0 \cdot n_t \cdot h_t;
\]

where \( h_t \) denotes the beginning-of-period \( t \) stock of human capital. The only way to increase this is to reduce the labor input in the production function (think of this as capturing the notion that the only way to expand one's skills is to withdraw from market production to enroll in school or in some other form of training.) The technology for increasing the stock of human capital is:

\[
    h_{t+1} = h_t + \delta (h_t \cdot n_t);
\]
The initial stock of human capital, \(h_0\), is given to the planner. The planner seeks to maximize discounted utility subject to the indicated constraints, by choice of \( f h_{t+1}; n_t; k_{t+1}; c_t; t = 0; 1; 2; \ldots\):

(i) Show that this planning problem has an alternative representation, in which the planner solves:

\[
\max_{f y_t; x_{t+1}} \sum_{t=0}^{\infty} \gamma^t h_t u(x_t; y_t; x_{t+1});
\]

where \(x_t = \frac{k_t}{h_t}\); \(y_t = \frac{h_{t+1}}{h_t}\): Display the exact functional form of \(u\); Also, display the constraints on \(x_{t+1}; y_t\):

(ii) Write the problem in recursive form. (Hint: use the logic in the hand-out and make use of the fact \(h_t = y_t \cdot (1+y_t)\), beyond the planner's control.)

(iii) Prove that there is a unique value function that solves the functional equation associated with (ii). (Hint: use the logic of the proof to Theorem 4.6.) Would you have had problems here if the restriction on \(\gamma\) had been \(\gamma < 1\) rather than \(0 < \gamma < 1\)?

2. Suppose a planner chooses to maximize, by choice of \(c_0; c_1; c_2; \ldots\); the following expression:

\[
u(c_0) + \pm^2 u(c_1) + \pm^2 u(c_2) + \cdots; u(c_t) = \log(c_t)\]

subject to

\[c_t = k_t \cdot c_{t+1}; 0 < \pm < 1; c_t; k_{t+1}, 0; k_0;\]

where \(0 < \pm < 1\): When \(\pm = 1\), this is the problem studied in exercises 2.2 and 4.9 in SL.

(i) Use the fact that \(k^0 = k_t\); solves the version of the problem with \(\pm = 1\) to establish that the solution to the problem with \(\pm \neq 1\) has the form:

\[k_1 = gk_0; k_{t+1} = \gamma k_t; t = 1; 2; \ldots\]

where \(g\) is a scalar. Derive an explicit formula relating \(g\) to the parameters of the model, \(\gamma; \pm\).
(ii) Is there a unique $k^*$ with the property $k_1^* = k^*$ as $t \to 1$ for all $k_0$? Display a formula relating $k^*$ to the parameters of the model.

(iii) Suppose $\gamma = 1 = 1.03$; $\alpha = 0.36$; $\pm = 0.8$: Suppose $k_0 = k^*$: Display the values of $k_0; k_1; k_2; k_3; k_4; k_5$ that solve the problem as of date zero.

(iv) Now suppose that when date 1 happens, the planner decides to re-optimize with respect to $k_2; k_3; \ldots$. The initial condition for this problem is $k_1$: the decision implemented by the planner last period. The planner’s preferences over $c_t$ are as follows:

$$u(c_1) + \pm u(c_2) + \pm^2 u(c_3) + \cdots$$

and the resource constraint is as before. What values will the planner choose for $k_1; k_2; k_3; k_4; k_5$? If the planner chooses to reoptimize in this way every period, to what value will $k_t$ actually tend?

(v) Are the values for $k_2; k_3; k_4; k_5$ chosen by the planner in date 1 the same as the values for these variables chosen in date 0? Why not? Because the chosen values for these variables disagree between time 0 and time 1, this problem is said to be time inconsistent. If $\pm$ had been set to one, we would not have had this problem. Why not?

(vi) Basically, the attitude of the planner is ‘I’m very impatient today (the discount rate from period 0 to period 1 is $\gamma = \pm$), but I’ll be less impatient tomorrow (the discount rate from period 1 to period 2 is $\gamma$), so I’ll consume a lot today and save a lot tomorrow.’ Such an attitude is not time consistent because when tomorrow rolls around the planner says the same thing. In the end, the planner just ends up with a low capital stock. This type of model has been used to explain the behavior of smokers, who resolve that ‘tomorrow I’ll quit smoking, but tonight I’ll just have one or two more’. Does the solution concept that we have used make any sense? Would a rational person make decisions in the time-inconsistent way described in (iv) and (v), or would they do something else? Answers to this question often involve posing the problem as a game between the planner in period $t$ and the planner in period $t+1$, and takes us beyond the scope of this course.

3. Let

$$D = \lim_{T \to 1} \frac{X}{T}^{-t}_t F_t;$$

where $0 < \gamma < 1$ and there exist finite $F^1 < F^2$ such that $F^1 < F_t < F^2$ for all $t$: Prove that $D$ exists.
4. Exercise 4.4 in SL. A few notes: (i) An assumption that differentiates this example from the previous ones is that \( X = f x_1; x_2; \ldots g \) is finite or countable. Previously, we have been assuming simply that \( X \) is continuous, i.e., \( X \subseteq \mathbb{R} \). For example, the finiteness assumption in the context of the growth model corresponds to requiring that the capital stock, \( k \), only take on values lying on a finite grid of, say, \( m \) numbers, \( K = f x_1; x_2; \ldots; x_m g \), and that the new capital decision similarly must lie on that grid, with the feasibility set being \( k^0 \leq k \leq k^0 + F(k; 1) + (1 - \beta)k \cap K \). Exercise 4.4 is of substantial practical significance, since it describes literally how dynamic programming techniques are applied in practice to solve models. The solution you get is actually only an approximation, since the underlying model of interest typically specifies that \( X \) is continuous. However, presumably the solution to the discrete model is very close to that of the continuous one when the points in \( X \) are very close together and \( m \) is very large. (ii) Note too, that the algorithm analyzed in this exercise is the same as the one you studied in question 1 in homework 2. Recall that algorithm converged in a finite number of steps (one, to be precise!), whereas when the algorithm based on the \( T \) operator alone was applied in class to solve the same model, we required literally an infinite number of steps to converge to the solution (verify this). (iii) Expansion on hint in 4.4 c: if a sequence of \( w_n \) is not only monotonically increasing, but also is bounded above (i.e., belongs to \( B(X) \)), then it converges. (iv) Another hint: though the contraction mapping theorem is not used to prove convergence of \( w_n \) it still plays an essential role in this question.

5. Exercise 5.1 a-e in SL, pages 104-105.