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D11-2, Winter 1995

MIDTERM EXAM

There are four questions. The total number of possible points is 100, and the number of points per question is given in parentheses. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have 2 hours. Good luck!

1. (30) The typical household can engage in two types of activities: producing current output and studying at home. Although time spent on studying at home sacrifices current production, it augments future output by increasing the household's future stock of human capital, k_{t+1}. The household has one unit of time available to split between home study and current production. Any given amount of human capital accumulation, k_{t+1}/k_t, leaves an amount of time, h_t, left over for producing current output, where $h_t = \phi(k_{t+1}/k_t)$. Here, ϕ is strictly decreasing, strictly concave, and continuously differentiable, with

$$\phi(1-\delta) = 1 \text{ for some } \delta \in (0,1),$$
$$\phi(1+\lambda) = 0 \text{ for some } \lambda > 0.$$

The variable, h_t, must satisfy $0 \leq h_t \leq 1$. This implies that the household cannot set k_{t+1}/k_t greater than $1+\lambda$ or less than $1-\delta$. See Figure 1 at the end of the exam for a graph of $h = \phi$.

A household's effective labor input into production is the product of its time and human capital: $h_t k_t$. Total output is related to effective labor input by

$$f(h_t k_t) = (h_t k_t)^{\alpha}, \quad \alpha \in (0,1).$$

The resource constraint for this economy is

$$c_t \leq f(h_t k_t),$$

and the initial level of human capital, k_0, is given. The utility value of a given sequence of consumption, c_t, is given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad \text{where } u(c_t) = c_t^{\sigma}/\sigma, \ \sigma < 0.$$

(a) Express the planning problem for this economy as a sequence problem (SP). Write out the associated functional equation (FE).

(b) Show that v(k) = Ak¾® is a solution to FE, and that the maximum is attained by the policy function, g(k) = µk for some (1 ¡ ±) < µ < (1 + ¸): (You may assume, without proof, that the maximum of the FE problem occurs in the interior of the relevant constraint set. Explain why this assumption is useful.)

(c) Suppose there are two separate economies, which di®er only in how patient households are. In the more patient economy the discount rate is ¯**e** and in the less patient economy, the discount rate is ¯ < ¯**e**: Show that in the economy with more patient households, the growth rate of human capital is greater.

2. (15) De¯nition: Let (S; ½) be a metric space and T : S ! S be a function mapping S into itself. T is a contraction mapping (with modulus ¯) if for some ¯ 2 (0; 1), ½(Tx; Ty) · ¯½(x; y); for all x; y 2 S:

   Prove the following theorem: Let X µ <ˡ; and let B(X) be a space of bounded functions f : X ! <; with the sup norm. Let T : B(X) ! B(X) be an operator satisfying

   ² f; g 2 B(X) and f(x) · g(x); for all x 2 X; implies (Tf)(x) · (Tg)(x); for all x 2 X;

   ² there exists some ¯ 2 (0; 1) such that

   [T(f + a)](x) · (Tf)(x) + ¯a; all f 2 B(X); a ¸ 0; x 2 X:

   Then T is a contraction mapping with modulus ¯:

3. (35) Consider the following two-sector economy. Consumption goods are produced using the following production function:

$$c_t = k_t^{µ} n_{1t}^{(1 ¡ µ)}; \; 0 < µ < 1:$$

   Capital goods depreciate completely in one period and are produced using labor only:

$$k_{t+1} = n_{2t}:$$

The total amount of labor in this economy is 1, so the following condition must be satisfied:

$$n_{1t} + n_{2t} = 1.$$

Let preferences be given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where u is strictly concave.

(a) Show that the problem has the following recursive representation:

$$v(k) = \max_{k' \in \Gamma(k)} F(k, k') + \beta v(k'), \text{ for all } k \in K,$$

and give an expression for F, Γ and K.

(b) Let

$$g(k) = \arg\max_{k' \in \Gamma(k)} F(k, k') + \beta v(k'), \text{ for all } k \in K.$$

The function, $g : K \to K$ is decreasing (don't confuse this with the one studied in class, which was increasing). Show this as rigorously as you possibly can, but be sure to take into account your time constraints. Make sure you at least explain, at a broad level, all the things that go into establishing this result. Then go into details. If you wish to appeal to a theorem, state it as precisely as you can.

4. (20) Let $X \subseteq \mathbb{R}^l$; let the correspondence $\Gamma : X \to X$ be nonempty and finite-valued; let $A = \{(x, y) \in X \times X : y \in \Gamma(x)\}$; let $F : A \to \mathbb{R}$ be a bounded function; and let $0 < \beta < 1$. Let $B(X)$ be the set of bounded functions $f : X \to \mathbb{R}$, with the sup norm. Define the operator T by

$$(Tf)(x) = \max_{y \in \Gamma(x)} [F(x, y) + \beta f(y)].$$

Let H be the set of functions $h : X \to X$ such that $h(x) \in \Gamma(x)$, all $x \in X$. For any $h \in H$, define the operator T_h on $B(X)$ by $(T_h f)(x) = F[x, h(x)] + \beta f[h(x)]$.

(a) Argue that for any $h \in H$; $T_h : B(X) \to B(X)$; and T_h has a unique fixed point $w \in B(X)$: (You may have to appeal to a theorem - if so, just state it as best as you can.)

(b) Let $h_0 \in H$ be given, and consider the following algorithm. Given h_n; let w_n be the unique fixed point of T_{h_n}: Given w_n; choose h_{n+1} so that $h_{n+1}(x) \in \arg\max_{y \in \Gamma(x)}[F(x; y) + \beta w_n(y)]$: Show that $w_0 \le Tw_0 \le w_1 \le Tw_1 \ldots:$