1. Answer for question 1.

(a) Market decentralizations.

i. Sequence of markets equilibrium. At each date, t, the household maximizes discounted utility from then on:

\[ \max_{j=t} \sum_{j} t u(c_j); \]

subject to a sequence of budget constraints:

\[ c_j + \pi_j \cdot r_j k_p + w_j n_j T_j + \frac{1}{s} j, t; \]

where \( w_j \) and \( r_j \) are market prices beyond the control of the household. The household uses its entire endowment of time for labor \( n \); because it does not value leisure. The firms choose \( n_t \) and \( k_{pt} \) such that pro\( ^\ast \)fits are maximized, where pro\( ^\ast \)fits are defined as follows:

\[ \frac{1}{s} = k_{gt} n_t (\alpha_i \alpha_k) \frac{k_{pt}}{k_{pt}} \frac{w_t n_t}{r_t}; \]

A sequence of markets equilibrium is a set of prices and quantities, \( f r_t; w_t; t \); 0g; \( f y_t; c_t; n_t; k_{pt}; k_{gt}; i_{pt}; k_{gt}; t \); 0g taxes, \( f T_t; t \); 0g; and pro\( ^\ast \)fits, \( f \frac{1}{s}; \) such that:

1. for each t; given taxes, pro\( ^\ast \)fits and prices, the quantities solve the household problem.

2. given the prices and sequence of government capital stocks, the quantities solve the rm problem, for all t.

2. given the quantities and a value of s, the government budget constraint is satis\( ^\ast \)ed for all t.

2 the resource constraint is satis\( ^\ast \)ed for all t.
ii. Date zero, Arrow-Debreu equilibrium. Let $f_{p,t}g$ denote the sequence of date $t$ consumption goods prices, denominated in date 0 consumption units. Let $f_{r,t}g$ and $f_{w,t}g$ denote the sequences of capital rental rates and wage rates, denominated in date $t$ consumption units. The household's budget constraint is:

$$
\prod_{t=0}^{\infty} p_t[c_t + i_{pt}] \cdot \prod_{t=0}^{\infty} p_t[r_t k_{pt} + w_t n_t] = T_t + \frac{1}{4}
$$

At date 0, the household enters all markets, and selects quantities to maximize utility.
Consider the firm problem. At date 0, the firm rents factors of production at all dates in order to solve

$$
\frac{1}{4} = \max_{f_{y,t}; k_{pt}; n_t; i_{gt}} \prod_{t=0}^{\infty} p_t[y_t + r_t k_{pt} + w_t n_t] ;
$$

subject to the production function, and the given sequences of prices and $k_{gt}$.

The government's budget constraint is:

$$
\prod_{t=0}^{\infty} p_t[i_{gt}] \cdot 0
$$

A date 0 Arrow-Debreu competitive equilibrium is a set of prices $f_{p,t}; r_t; w_t; T_t; i_{gt}$; quantities, $f_{y,t}; c_t; n_t; k_{pt}; k_{gt}; i_{pt}; i_{gt}; T_t; i_{gt}$; 0g taxes, $fT_t; t$; 0g; and a level of profit $\frac{1}{4}$ such that

2 given the prices and level of profits, the quantities solve the household problem.
2 given the prices and sequence of government capital stocks, the quantities solve the firm problem.
2 given the quantities and a value of $s$, the government budget constraint is satisfied.
2 the resource constraint is satisfied for all $t$.

iii. A recursive competitive. First, define the household problem. To define the problem, for the household to know three numbers: $K_p; K_g$; and $k_p$; where the first two objects are the economy-wide stocks of private and government capital (note
the slight switch in notation regarding government capital), and the individual household's stock of capital. They also need four functions, \( \frac{1}{4}(K_p; K_g); r(K_p; K_g); w(K_p; K_g); I(K_p; K_g); \) which are the level of profits, competitive rental rate and wage rate, the level of taxes and economy-wide average level of investment. The problem is to choose an investment level, \( i(K_p; K_g; k_p); \) and a level of employment \( n(K_p; K_g; k_p) \) to solve

\[
v(K_p; K_g; k_p) = \max_{i_g, n} u(c) + \frac{1}{n} v(K_p; K_g; k_p; k^0_p); \\
subject to 0 \cdot n \cdot n; c \geq 0; k^0_p = (1 + i_p); and
\]

\[
K^0_g = (1 + i_p) + I(K_p; K_g); c + i_g \cdot r(K_p; K_g)k_p + w(K_p; K_g)n; T(K_p; K_g);
\]

The second problem is to solve

\[
\frac{1}{4}(K_p; K_g) = \max_{K_p; K_g} \frac{1}{4} \left( K_p; K_g \right); r(K_p; K_g); w(K_p; K_g); T(K_p; K_g); I(K_p; K_g); i(K_p; K_g; k_p); n(K_p; K_g; k_p); v(K_p; K_g; k_p);
\]

which satisfy:

\[ 2 \] given \( \frac{1}{4} r; w; T; i; \); the functions \( v; i \) and \( n \) solving the household problem, for each \( k_p; K_p; K_g; \)

\[ 2 \] for each \( K_p; K_g \); the quantities \( k_p; \frac{1}{4}(K_p; K_g) \) and \( n(K_p; K_g; k_p) \) solve the second problem, given \( r(K_p; K_g); w(K_p; K_g); \)

\[ 2 \] the consistency conditions are satisfied

\[ 2 \] the resource constraint is satisfied, \( c + i_g + i_p \cdot y; \) for all \( K_p; K_g; \)
(b) the first order condition for the household is

$$u_{ct} = ^{\prime}u_{ct+1}[r_{t+1} + 1 \ i \ \mp];$$

and the first order condition for the firm is

$$f_{k_{pt};t+1} = r_{t+1};$$

where \( f_{k_{pt};t+1} \) is the marginal product of private capital. Combining these, and taking functional forms into account:

$$\mu_{ct+1} \frac{\partial}{\partial c_t} = - \left[ \frac{\partial}{\partial k_{glt+1}} \frac{1}{k_{pt+1}} + 1 \ i \ \mp \right];$$

Let \( g_c \) denote the gross growth rate of consumption in a balanced growth path. Then,

$$\left( g_c \right)^{0} = - \left[ \partial (ns) \right]^{1 \ \otimes} + 1 \ i \ \mp;$$

Suppose \( g_c \) corresponds to some given positive net growth rate, i.e., \( g_c > 1 \). Then,

$$s = \frac{1}{n} \left( \frac{1}{n} \frac{g_c^{0}}{\partial^{\otimes}} + \frac{1}{\partial^{\otimes}} + 1 \right);$$

The number in square brackets is positive, so that \( s \) is well defined. Thus the Euler equation is consistent with constant consumption growth in steady state. To fully answer the question, we need to establish (i) that the other equations - the household budget equation and the resource constraint - are also satisfied with a constant consumption growth rate and (ii) that the other quantity variables display positive growth too. Let \( g_g \) and \( g_p \) denote the gross growth rates of government and private capital, respectively. Then, the government's policy for choosing \( k_{gt} \) implies:

$$g_g = g_p = g;$$

say. Note that output can be written

$$k_{gt}^{(1 \ \otimes)} k_{pt}^{\otimes} n^{(1 \ \otimes)} = k_{gt} (k_{pt} = k_{gt})^{\otimes} n^{(1 \ \otimes)} = k_{gt} s^{\otimes} n^{(1 \ \otimes)};$$

Divide the resource constraint by \( k_{gt} \):

$$\frac{c_t}{k_{gt}} + g_{t+1} i (1 \ i \ \mp) + g_{t+1} i (1 \ i \ \mp) = s^{\otimes} n^{(1 \ \otimes)};$$
So, in a constant growth steady state (i.e., $g_{t+1} = g$; constant) the consumption to public capital ratio is a constant, equal to the following:

$$s^\circ n^{(1) \circ} + (1 \ i \ \pm g) + (1 \ i \ \pm p) + 2g$$

But, the consumption to public capital ratio being constant implies:

$$g_c = g$$

The household budget constraint is trivially satisfied, since it is equivalent with the resource constraint given the first order conditions of rms, linear homogeneity of the production function with respect to rms' choice variables, and the government budget constraint.

(c) The planner’s problem is: choose $c_t; k_{gt+1}; k_{pt+1}; t = 0$ to maximize discounted utility. After substituting out consumption using the resource constraint, the problem becomes:

$$\max_{f_{k_{gt+1}k_{pt+1}g_t=0}} \sum_{t=0}^{\infty} u^{(1) \circ} [n^{(1) \circ} k_{gt+1}^{(1) \circ} k_{pt+1}^{(1) \circ} + (1 \ i \ \pm g) k_{gt} + (1 \ i \ \pm p) k_{pt}]$$

subject to the object in square brackets (consumption) being non-negative at all dates, and to $k_{gt}, k_{pt} \geq 0$: The planner’s first order conditions are:

$$u_{ct} = -u^{(1) \circ}[f_{k_{gt+1}k_{pt+1}g_t=0} + 1 \ i \ \pm p]$$

for $t = 0; 1; 2; \ldots$. With the functional forms:

$$\frac{\mu c_{t+1}}{c_t} = -[\circ k_{gt+1}^{(1) \circ} \frac{n^{(1) \circ}}{k_{pt+1}^{(1) \circ}} + 1 \ i \ \pm p]$$

$$\frac{\mu c_{t+1}}{c_t} = -[\circ (k_{gt+1})^{(1) \circ} \frac{n^{(1) \circ}}{(k_{pt+1})^{(1) \circ}} + 1 \ i \ \pm p]$$

Substituting out consumption using the resource constraint, these two equations represent a vector difference equation in $k; k^{(1) \circ}, k^{(1) \circ}$. 
where $k = [k_g \ k_p]^T$. There are many solutions to this equation that are consistent with the given initial condition, $k_0 = [k_{g0} \ k_{p0}]$: One can construct the whole family of solutions by indexing them by $k_1$: different values of $k_1$ give rise, by iterating on the euler equation, to different sequences of capital. Not all are optimal. Only the one solution that also satisfies the transversality condition is optimal. Thus, satisfying the Euler equation is not sufficient for an optimum.

(d) Setting $\delta = 1$ and equating the planner’s two first order conditions, we get:

$$\bar{A} \left\{ \delta \frac{n_k t+1}{k_{pt+1}} + 1 \right\} = \bar{A} \left\{ \delta \frac{k_{pt+1}}{k_{gt+1}} + 1 \right\};$$

which requires that $\frac{k_{pt+1}}{k_{gt+1}}$ be a particular constant for $t = 0; 1; ::::$. Call this constant $s^\infty$. By setting $s = s^\infty$ the government cannot do better, since this achieves the planner’s optimum.