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FINAL EXAM ANSWERS

1. Answer for question 1.

   (a) Market decentralizations.

      i. Sequence of markets equilibrium. At each date, t, the household maximizes discounted utility from then on:

$$\sum_{j=t}^{\infty} \beta^{j-t} u(c_j);$$

subject to a sequence of budget constraints:

$$c_j + i_{pj} \leq r_j k_{pj} + w_j n - T_j + \pi_j;\ j \geq t;$$

where w_j and r_j are market prices beyond the control of the household. The household uses its entire endowment of time for labor effort, n; because it does not value leisure. The firms choose n_t and $k_{p,t}$ such that profits are maximized, where profits are defined as follows:

$$\pi_t = k_{g,t}^{\gamma} n_t^{(1-\alpha)} k_{pt}^{\alpha} - w_t n_t - r_t k_t.$$

A sequence of markets equilibrium is a set of prices and quantities, $\{r_t, w_t; t \geq 0\}$; $\{y_t, c_t, n_t, k_{pt}, k_{gt}, i_{pt}, i_{gt}; t \geq 0\}$ taxes, $\{T_t; t \geq 0\}$; and profits, $\{\pi_t\}$; such that

- for each t; given taxes, profits and prices, the quantities solve the household problem.
- given the prices and sequence of government capital stocks, the quantities solve the firm problem, for all t.
- given the quantities and a value of s, the government budget constraint is satisfied for all t.
- the resource constraint is satisfied for all t.

ii. Date zero, Arrow-Debreu equilibrium. Let $\{p_t\}$ denote the sequence of date t consumption goods prices, denominated in date 0 consumption units. Let $\{r_t\}$ and $\{w_t\}$ denote the sequences of capital rental rates and wage rates, denominated in date t consumption units. The household's budget constraint is:

$$\sum_{t=0}^{\infty} p_t[c_t + i_{pt}] \le \sum_{t=0}^{\infty} p_t[r_r k_{pt} + w_t n_t - T_t] + \pi.$$

At date 0, the household enters all markets, and selects quantities to maximize utility.

Consider the firm problem. At date 0, the firm rents factors of production at all dates in order to solve

$$\pi = \max_{\{y_t; k_{pt}; n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} p_t[y_t - r_t k_{pt} - w_t n_t];$$

subject to the production function, and the given sequences of prices and k_{gt}.

The government's budget constraint is:

$$\sum_{t=0}^{\infty} p_t[i_{gt} - T_t] \le 0.$$

A date 0 Arrow-Debreu competitive equilibrium is a set of prices $\{p_t; r_t; w_t; t \ge 0\}$; quantities, $\{y_t; c_t; n_t; k_{pt}; k_{gt}; i_{pt}; i_{gt}; t \ge 0\}$ taxes, $\{T_t; t \ge 0\}$; and a level of profits, π; such that

- given the prices and level of profits, the quantities solve the household problem.
- given the prices and sequence of government capital stocks, the quantities solve the firm problem
- given the quantities and a value of s, the government budget constraint is satisfied.
- the resource constraint is satisfied for all t.

iii. A recursive competitive. First, define the household problem. To define the problem, for the household to know three numbers: K_p; K_g; and k_p; where the first two objects are the economy-wide stocks of private and government capital (note the slight switch in notation regarding government capital), and the individual household's stock of capital. They also need four functions, $\pi(K_p, K_g)$, $r(K_p, K_g)$, $w(K_p, K_g)$, $T(K_p, K_g)$, $I(K_p, K_g)$, which are the level of profits, competitive rental rate and wage rate, the level of taxes and economy-wide average level of investment. The problem is to choose an investment level, $i(K_p, K_g, k_p)$, and a level of employment $n(K_p, K_g, k_p)$ to solve

$$v(K_p, K_g, k_p) = \max_{i_g, \tilde{n}} u(c) + \beta v(K_p', K_g', k_p'),$$

subject to $0 \le \tilde{n} \le n$, $c \ge 0$, $k_p' \ge 0$, $k_p' = (1-\delta)k_p + i_p$, and

$K_g' = (1-\delta)K_g + I(K_p, K_g)$, $c + i_g \le r(K_p, K_g)k_p + w(K_p, K_g)\tilde{n} - T(K_p, K_g)$.

The firm problem is to solve

$$\pi(K_p, K_g) = \max_{\tilde{k}_p, \tilde{n}} y - r(K_p, K_g)\tilde{k}_p - w(K_p, K_g)\tilde{n}.$$

The government has to satisfy a period-by-period budget constraint, $i_g = T(K_p, K_g)$.
There are two consistency conditions:

$$k_p = K_p, \text{ and } i(K_p, K_p, K_g) = I(K_p, K_g).$$

The first of these says that everyone's individual stock of capital has to equal the aggregate (economy-wide average) stock. The second says that everyone's individual investment decision has to correspond to the economy wide average.
A recursive competitive equilibrium is a set of functions: $\pi(K_p, K_g)$, $r(K_p, K_g)$, $w(K_p, K_g)$, $T(K_p, K_g)$, $I(K_p, K_g)$, $i(K_p, K_g, k_p)$, $n(K_p, K_g, k_p)$, $v(K_p, K_g, k_p)$, which satisfy:

- given π, r, w, T, I, the functions v, i and n solve the household problem, for each k_p, K_p, K_g.
- for each K_p, K_g, the quantities K_p, $\pi(K_p, K_g)$ and $n(K_p, K_g, K_p)$ solve the firm problem, given $r(K_p, K_g)$, $w(K_p, K_g)$.
- the consistency conditions are satisfied
- the resource constraint is satisfied, $c + i_g + i_p \le y$, for all K_p, K_g.

(b) the first order condition for the household is

$$u_{c,t} = \beta u_{c,t+1}[r_{t+1} + 1 - \delta_p],$$

and the firm sets $f_{k_p,t+1} = r_{t+1}$, where $f_{k_p,t+1}$ is the marginal product of private capital. Combining these, and taking functional forms into account:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\gamma} = \beta\left[\alpha\left(\frac{nk_{g,t+1}}{k_{p,t+1}}\right)^{(1-\alpha)} + 1 - \delta_p\right].$$

Let g_c denote the gross growth rate of consumption in a balanced growth path. Then,

$$(g_c)^{\gamma} = \beta[\alpha(ns)^{(1-\alpha)} + 1 - \delta_p].$$

Suppose g_c corresponds to some given positive net growth rate, ie., $g_c > 1$. Then,

$$s = \frac{1}{n}\left\{\frac{1}{\alpha}\left[\frac{g_c^{\gamma}}{\beta} + \delta_p - 1\right]\right\}^{\frac{1}{1-\alpha}}.$$

The number in square brackets is positive, so that s is well defined. Thus the Euler equation is consistent with constant consumption growth in steady state. To fully answer the question, we need to establish (i) that the other equations - the household budget equation and the resource constraint - are also satisfied with a constant consumption growth rate and (ii) that the other quantity variables display positive growth too. Let g_g and g_p denote the gross growth rates of government and private capital, respectively. Then, the government's policy for choosing $k_{g,t}$ implies:

$$g_g = g_p = g,$$

say. Note that output can be written

$$k_{gt}^{(1-\alpha)}k_{pt}^{\alpha}n^{(1-\alpha)} = k_{gt}(k_{pt}/k_{gt})^{\alpha}n^{(1-\alpha)} = k_{gt}s^{\alpha}n^{(1-\alpha)}.$$

Divide the resource constraint by k_{gt}:

$$\frac{c_t}{k_{gt}} + g_{t+1} - (1 - \delta_g) + g_{t+1} - (1 - \delta_p) = s^{\alpha}n^{(1-\alpha)}.$$

So, in a constant growth steady state (i.e., $g_{t+1} = g$; constant) the consumption to public capital ratio is a constant, equal to the following:

$$s^{\alpha} n^{(1-\alpha)} + (1 - \delta_g) + (1 - \delta_p) + 2g.$$

But, the consumption to public capital ratio being constant implies:

$$g_c = g.$$

The household budget constraint is trivially satisfied, since it is equivalent with the resource constraint given the first order conditions of firms, linear homogeneity of the production function with respect to firms' choice variables, and the government budget constraint.

(c) The planner's problem is: choose $c_t, k_{g,t+1}, k_{p,t+1}$, $t \geq 0$ to maximize discounted utility. After substituting out consumption using the resource constraint, the problem becomes:

$$\max_{\{k_{g,t+1},k_{p,t+1}\}} \sum_{t=0}^{\infty} \beta^t u[k_{gt}^{(1-\alpha)} n^{(1-\alpha)} k_{pt}^{\alpha} + (1-\delta_g)k_{g,t} + (1-\delta_p)k_{p,t} - k_{p,t+1} - k_{g,t+1}],$$

subject to the object in square brackets (consumption) being nonnegative at all dates, and to $k_{g,t}, k_{p,t} \geq 0$. The planner's first order conditions are:

$$\begin{aligned} u_{c,t} &= \beta u_{c,t+1}[f_{k_{p,t+1}} + 1 - \delta_p] \\ u_{c,t} &= \beta u_{c,t+1}[f_{k_{g,t+1}} + 1 - \delta_g], \end{aligned}$$

for $t = 0, 1, 2, \ldots$. With the functional forms:

$$\begin{aligned} \left(\frac{c_{t+1}}{c_t}\right)^{\gamma} &= \beta\left[\alpha k_{g,t+1}^{\gamma} \left(\frac{n}{k_{p,t+1}}\right)^{(1-\alpha)} + 1 - \delta_p\right] \\ \left(\frac{c_{t+1}}{c_t}\right)^{\gamma} &= \beta[\gamma (k_{g,t+1})^{\gamma-1} n^{(1-\alpha)} (k_{p,t+1})^{\alpha} + 1 - \delta_g]. \end{aligned}$$

Substituting out consumption using the resource constraint, these two equations represent a vector difference equation in k, k', k'', where $k = [k_g \; k_p]'$. There are many solutions to this equation that are consistent with the given initial condition, $k_0 = [k_{g,0} \; k_{p,0}]$. One can construct the whole family of solutions by indexing them by k_1: different values of k_1 give rise, by iterating on the euler equation, to different sequences of capital. Not all are optimal. Only the one solution that also satisfies the transversality condition is optimal. Thus, satisfying the Euler equation is not sufficient for an optimum.

(d) Setting $\gamma = 1 - \alpha$ and equating the planner's two first order conditions, we get:

$$\begin{aligned} &\beta\left[\alpha \left(\frac{n k_{g,t+1}}{k_{p,t+1}}\right)^{(1-\alpha)} + 1 - \delta_p\right] \\ = \; &\beta\left[(1-\alpha) n^{(1-\alpha)} \left(\frac{k_{p,t+1}}{k_{g,t+1}}\right)^{\alpha} + 1 - \delta_g\right]; \end{aligned}$$

which requires that $\frac{k_{p,t+1}}{k_{g,t+1}}$ be a particular constant for $t = 0, 1, \ldots$. Call this constant s^*. By setting $s = s^*$ the government cannot do better, since this achieves the planner's optimum.