

Homework #4  
 Economics D11-2  
 Due Wednesday, January 31.

1. In this question, you are asked to prove the partial converse of theorem 4.15 in SL: Let  $x_1^a; x_2^a; x_3^a; \dots$  solve the sequence problem, and suppose  $x_t^a$  is interior, that is,  $x_t^a \in \text{int}(X)$ ;  $x_{t+1}^a \in \text{int}(j(x_t^a))$  for  $t = 0; 1; \dots$ . Also, suppose A4:3; A4:7 and A4:9 are satisfied and, for simplicity,  $l = 1$ . Then, the following must be true:

$$F_2(x_t^a; x_{t+1}^a) + \beta F_1(x_{t+1}^a; x_{t+2}^a) = 0; \quad t = 0; 1; 2; \dots$$

The purpose of this question is to get you to prove this proposition using a variational argument.

- (a) Consider a particular family of variations on the optimal plan:  $x_1^a; x_2^a + \epsilon; x_3^a; \dots$ , where  $|\epsilon| < \delta$  and  $\delta > 0$ . Show that, for  $\delta$  small enough, every element in this family of variations is feasible. To illustrate the importance of continuity of  $j$  here, show that if  $j$  fails to be lower hemi-continuous, this statement need not be true.
- (b) Let

$$S(\epsilon) = F(x_0^a; x_1^a) + \beta F(x_1^a; x_2^a + \epsilon) + \beta^2 F(x_2^a + \epsilon; x_3^a) + \sum_{t=3}^{\infty} \beta^t F(x_t^a; x_{t+1}^a);$$

Show that  $S(\epsilon)$  is a strictly concave function of  $\epsilon$  in the range,  $|\epsilon| < \delta$  for some  $\delta > 0$ .

- (c) Prove the proposition.
- (d) State and prove the analogous result for the economy with uncertainty in question 2 (b) of Homework 2.
2. (Human capital) Consider the problem of a planner who seeks to maximize

$$\sum_{t=0}^{\infty} \beta^t [c_t^{\otimes}]; \quad 0 < \beta < 1;$$

subject to the following restrictions. The resource constraint is:

$$c_t + k_{t+1} \beta (1 - \delta) k_t = k_t^\mu n_t^{(1-\mu)}$$

where  $k_t$  denotes beginning-of-period  $t$  physical capital, and  $n_t$  denotes labor, measured in efficiency units (as opposed to numbers of people, or time). At date 0,  $k_0$  is given. Also, the following nonnegativity constraints must be respected:

$$c_t \geq 0; k_{t+1} \geq (1 - \delta)k_t; t = 0; 1; \dots$$

The constraints on labor are:

$$0 \leq n_t \leq h_t;$$

where  $h_t$  denotes the beginning-of-period  $t$  stock of human capital. The only way to increase this is to reduce the labor input in the production function (think of this as capturing the notion that the only way to expand one's skills is to withdraw from market production to enroll in school or in some other form of training.) The technology for increasing the stock of human capital is:

$$h_{t+1} = h_t + \delta(h_t - n_t);$$

and

$$\delta; \beta > 0 \text{ and } \beta(1 - \delta) < 1;$$

The initial stock of human capital,  $h_0$ , is given to the planner. The planner seeks to maximize discounted utility subject to the indicated constraints, by choice of  $\{h_{t+1}; n_t; k_{t+1}; c_t; t = 0; 1; 2; \dots; g\}$ :

- (a) Show that this planning problem has an alternative representation, in which the planner solves:

$$\max_{\{y_t; x_{t+1}\}_{t=0}^g} \sum_{t=0}^g \beta^t u(x_t; y_t; x_{t+1});$$

where  $x_t = \frac{k_t}{h_t}$ ;  $y_t = \frac{h_{t+1}}{h_t}$ . Display the exact functional form of  $u$ . Also, display the constraints on  $x_{t+1}; y_t$ :

- (b) Write the problem in recursive form. (Hint: use the logic in the handout and make use of the fact  $h_t = y_{t-1}y_{t-2}\dots y_0h_0$ ; where  $h_0$  is a constant, beyond the planner's control.)

- (c) Prove that there is a unique value function that solves the functional equation associated with (b). (Hint: use the logic of the proof to Theorem 4.6.) Would you have had problems here if the restriction on  $\beta$  had been  $\beta < 1$  rather than  $0 < \beta < 1$ ?
3. Exercise 4.4 in SL. (i) Note the assumption that  $X = \{x_1, x_2, \dots\}$  is finite or countable. Previously, we have been assuming simply that  $X$  is continuous, i.e.,  $X \subseteq \mathbb{R}^l$ . For example, the finiteness assumption in the context of the growth model corresponds to requiring that the capital stock,  $k$ , only take on values lying on a finite grid of, say,  $m$  numbers,  $K = \{x_1, x_2, \dots, x_m\}$ , and that the new capital decision similarly must lie on that grid, with the feasibility set being  $k^0 \geq 0, k = f_0 \cdot k^0 \cdot F(k; 1) + (1 - \delta)k \in K$ : Exercise 4.4 is of substantial practical significance, since it describes literally how dynamic programming techniques are applied in practice to solve models. The solution you get is actually only an approximation, since the underlying model of interest typically specifies that  $X$  is continuous. However, presumably the solution to the discrete model is very close to that of the continuous one when the points in  $X$  are very close together and  $m$  is very large. (ii) Expansion on hint in 4.4 c: if a sequence of  $w_n$  is not only monotonically increasing, but also is bounded above (i.e., belongs to  $B(X)$ ), then it converges. (iv) Another hint: though the contraction mapping theorem is not used to prove convergence of  $w_n$  it still plays an essential role in this question.