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1 Co si er he ro le of n i di id al er on ho ho se $c_0, c_1, c_2, \ldots,$ to maximize utility:

$$u(c_0) + \delta[\beta u(c_1) + \beta^2 u(c_2) + \ldots], \; u(c_t) = \log(c_t)$$

subject to the following budget constraint:

$$c_t = k_t^{\alpha} - k_{t+1}, \; 0 < \alpha < 1, \; c_t, k_{t+1} \geq 0, \; k_0,$$

where $0 < \beta < 1, 0 < \delta < 1.$ When $\delta = 1,$ this is the problem studied in exercises 2.2 and 4.9 in SL.

(a) Verify that the solution to this problem has the form:

$$k_1 = g k_0^{\alpha}, \; k_{t+1} = \beta \alpha k_t^{\alpha}, \; t = 1, 2, \ldots,$$

where $g$ is a scalar. Derive an explicit formula relating $g$ to the parameters of the model, $\beta, \alpha, \delta.$

(b) Is there a unique $k^*$ with the property $k_t \rightarrow k^*$ as $t \rightarrow \infty$ for all $k_0$? Display a formula relating $k^*$ to the parameters of the model.

(c) Suppose $\beta = 1/1.03, \alpha = .36, \delta = .8.$ Suppose $k_0 = k^*.$ Display the values of $k_0, k_1, k_2, k_3, k_4, k_5$ that solve the problem as of date zero.

(d) Now suppose that at date 1, after $k_1$ has been chosen, the household decides to reoptimize with respect to $k_2, k_3, \ldots.$ The initial condition for this problem is $k_1$. The individual's preferences over $c_t, t \geq 1$ are as follows:

$$u(c_1) + \delta[\beta u(c_2) + \beta^2 u(c_3) + \ldots]$$

and the budget constraint is as before. What values will the household choose for $k_2, k_3, k_4, k_5$? If the household chooses to reoptimize in this way every period, to what value will $k_t$ actually tend?

(e) Are the values for $k_2; k_3; k_4; k_5$ chosen by the household in date 1 the same as the values for these variables chosen in date 0? Why not? Because the chosen values for these variables differs between time 0 and time 1, this problem is said to be time inconsistent. If $\delta$ had been set to one, we would not have had this problem. Why not?

(f) Basically, the attitude of the household is `I'm very impatient today (the discount rate from period 0 to period 1 is $\beta\delta$), so today I won't save much, but I'll be less impatient tomorrow (the discount rate from period 1 to period 2 is $\beta$), so starting tomorrow I'll save a lot.' Such a plan is not time consistent because when tomorrow rolls around the household says the same thing: `today I won't save much, but starting tomorrow I'll save a lot'. The actions through time of this person, who obviously does not have much insight, turn out to coincide with those of a person with plenty of insight, but who has no self control: At date t it chooses $k_{t+1}$ to maximize utility, while treating its future self as an independent person who will then do whatever is best for it, given whatever capital stock it inherits from the past. Show that a policy, $k_{t+1} = gk_t^{\alpha}$; $t = 0; 1; 2; \ldots;$ where g coincides with the g above is a `solution': this policy has the property that it solves each date t self's maximization problem, conditional on the (correct) conjecture that all future selfs will follow this policy rule too. What is the date 0 present discounted utility for such a household, given $k_0 = k^*$?

(g) Now consider a household with self-control. It computes the $k_1; k_2; k_3; \ldots$ choices that are optimal as of date 0 and then sticks to its decision without fail. What is the date 0 present discounted utility for such a household, given $k_0 = k^*$?

(h) What constant fraction of each period's consumption would a household with no self control be willing to sacrifice at date zero, to acquire self control? (That is, if $c_0^*; c_1^*; c_2^*; \ldots$ is the consumption plan of the household with self control, and $c_0; c_1; c_2; \ldots;$ is the consumption plan of the household with no self control, what is the value of $\lambda$ such the present discounted utility of the consumption stream $(1 - \lambda)c_t^*$ coincides with the discounted utility enjoyed by the household with no self control?) Laibson has recently proposed using this type of utility function to help explain the variety of things that people and societies do to commit themselves to a high savings plan. The parameter $\lambda$ is a measure of the value of commitment in this context.

2. Exercise 6.7, parts (e)-(f) in S-L, page 158.

3. Consider a version of the endogenous growth model in exercise 5.8 (b) in S-L, with $u(c) = c^{\sigma}/\sigma$, $\sigma < 0$ (note: this differs from S-L, where $\sigma < 1$) and $\beta(\theta^{\alpha\sigma}) < 1$, for all $1 - \delta \leq \theta \leq 1 + \lambda$. Also, $0 < \lambda < 1$, $0 < \delta < 1$. The sequence representation of that problem is:

$$\max_{\{k_{t+1};t=0,1,2,...\}} \sum_{t=0}^{\infty} \beta^t \left[ k_t \phi\left(\frac{k_{t+1}}{k_t}\right)\right]^{\sigma\alpha} / \sigma,$$

subject to the given value of $k_0$ and $1 - \delta \leq k_{t+1}/k_t \leq 1 + \lambda$, for $t = 0, 1, ...$

(a) Show that the solution to this problem coincides with the solution to:

$$\begin{aligned} v = & \max_{\{\theta_t;t=0,1,...\}} \{\phi(\theta_0)^{\alpha\sigma}/\sigma + \beta\,[\theta_0\phi(\theta_1)]^{\alpha\sigma}/\sigma \\ & + \beta^2\,[\theta_0\theta_1\phi(\theta_2)]^{\alpha\sigma}/\sigma + \beta^3\,[\theta_0\theta_1\theta_2\phi(\theta_3)]^{\alpha\sigma}/\sigma + ...\} \end{aligned}$$

and that $-\infty < v < 0$. Here, $\theta_t = k_{t+1}/k_t$, $t = 0, 1, 2, ...$ .

(b) Argue that $v$ satisfies:

$$v = \max_{1-\delta \leq \theta \leq 1+\lambda} \phi(\theta)^{\alpha\sigma}/\sigma + \beta\,(\theta^{\alpha\sigma})\,v.$$

(c) Prove (using a suitably modified version of the proof of theorem 4.6) that there exists a unique $v$ that solves the above equation. (The fact that $u(c)$ is not defined at $c = 0$ causes a problem here. Is there an easy way out?) Let

$$\theta^* = \arg\max_{1-\delta \leq \theta \leq 1+\lambda} \phi(\theta)^{\alpha\sigma}/\sigma + \beta\,(\theta^{\alpha\sigma})\,v.$$

i. Argue that $\theta_t = \theta^*$, for $t = 0, 1, 2, \ldots$ solves the sequence problem (this argument need not be totally rigorous).

ii. Show that $\theta^*$ is increasing in $\beta$ (you may assume that $\theta^*$ is interior, i.e., $1 - \delta < \theta^* < 1 + \lambda$). Does this result make economic sense?