Homework 7
Due Friday, March 1.
D11-2, Winter 1996
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Homework/Tutorial on Endogenous Growth Models

1. Consider the following problem for a household:

$$U = \max_{\{c(s^t), l(s^t), k(s^t)\}} \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) u(c(s^t), l(s^t)), \tag{1}$$

subject to the budget constraint:

$$c(s^t) + p(s^t)I(s^t) \leq w(s^t)l(s^t) + r(s^t)k(s^t), \text{ all } s^t, \text{ all } t \geq 0, \tag{2}$$

where the (strictly positive) prices, $\{p(s^t), w(s^t), r(s^t), t \geq 0\}$, are taken as given by the household. Here, $0 \leq l(s^t) \leq 1$ denotes labor effort in s^t, and s^t is the history notation defined in a previous homework, and in class. To visualize the household's problem, imagine a tree, with nodes indexed by s^t for $t = 0, 1, ...,$ and prices written on each node. The household's task is to select a level of consumption, labor effort and investment, $I(s^t)$, for each node, to maximize present discounted utility, subject to the budget constraint being satisfied on each node, and subject to the given prices posted on each node. Investment and capital are related as follows:

$$k(s^t) = (1 - \delta)k(s^{t-1}) + I(s^t), \tag{3}$$

where $k(s^t)$ is the end-of-period stock of capital at node s^t. The utility function, u, satisfies the assumptions on the 'Canonical Model' handout. There is an important technical issue here regarding the finiteness of achievable discounted utility. You may ignore this technical issue by simply assuming that U, the best level of utility achievable by the household, is finite.[1]

[1] The problem here derives from the fact that the household's objective is an infinite sum, and so may not be finite. If 'infinite' utility were achievable, there might not be a well defined maximum. The assumption, $U < \infty$ is an implicit restriction on the sequence of prices faced by the household.

In chapter 4 there is a theorem which displays a particular set of Euler equations and a transversality condition. The theorem shows that if a set of feasible allocations satis¯es these equations, then the allocations are optimal. In a previous homework, you proved the analogous result for the case of uncertainty. Prove the same su±ciency result for the household problem. (Hint: here, `feasibility' means that the budget constraint is satis¯ed at all nodes. Also, recall that the core of the proof rides on exploiting a particular property of a ¯rst order Taylor series expansion of a strictly concave function.)

2. This is an exercise involving an endogenous growth model. A basic puzzle confronted by the growth literature is: `why doesn't the rate of return on capital go to zero eventually, at which point only maintenance investment is done, and no further e®orts are made to expand the capital stock?' Economists even as late as J.M. Keynes, took it for granted that diminishing returns to capital investment would eventually kick in, inevitably bringing the growth process to a halt (just as the one-sector `growth' model of Chapter 2 in S-L predicts). But, confounding the prediction of Keynes and many others, the rate of return of capital shows no signs of falling, and growth continues. There are now a variety of models incorporating various factors that can account for the failure of the rate of return on capital to drop, despite the continued increase in the stock of capital. In previous homeworks you have seen two such models. There, the economic decision to improve human capital continually pushes up the production function, preventing the return to investment from dropping. You have also seen the Solow model (the `Japan model') which posits that the production function shifts up for reasons that are exogenous to the saving and investment decision. Two additional types of growth model are considered in this homework. This question considers a model which posits that the production of new investment goods is heavily capital-intensive, and does not involve diminishing returns. (See Sergio Rebelo, `Long-run Policy Analysis and Long-Run Growth,' Journal of Political Economy, June 1991, for a careful exposition of this line of thinking.)

   The model in this homework is not just designed to articulate a particular view about what keeps economies growing. It is also confronts the empirical fact that the cost of new investment goods (particularly business equipment and household durables, but also structures) has been falling rapidly relative to the cost of consumption goods. The model, in effect, takes the position that this observation is the key to understanding why growth seems to be a permanent state of affairs.

The economy has a large number of identical agents, each having the preferences given in (??), with

$$u(c; l) = \begin{cases} [c(1-l)^{\psi}]^{(1-\sigma)}/(1-\sigma), & \text{for } \sigma > 0,\ \sigma \neq 1 \\ \log(c) + \psi \log(1-l), & \text{for } \sigma = 1. \end{cases} \tag{4}$$

The assumption that u is strictly increasing in c and strictly decreasing in l, and strictly concave corresponds to:

$$\sigma > 0, \psi > 0, \psi(1-\sigma) < 1, \sigma > \psi(1-\sigma).$$

There is no uncertainty in this economy. There are two separate technologies: one for producing the consumption good, c_t, and one for producing the investment good, I_t. The first technology is:

$$c_t \leq A k_{ct}^{\alpha} l_t^{(1-\alpha)}, \tag{5}$$

where A is a positive constant, k_{ct} is capital used in the consumption sector, and $0 < \alpha < 1$. The technology for producing investment goods is:

$$I_t = b k_{It}, \tag{6}$$

where k_{It} is the stock of capital used in the investment sector and b is a positive constant. At any particular point of time, t, the aggregate stock of capital, k_t, is given. There are no restrictions on how that capital may be allocated between the two sectors, subject to:

$$k_t = k_{ct} + k_{It},\ k_{ct} \geq 0,\ k_{It} \geq 0. \tag{7}$$

The model is like the one analyzed in the previous homework (question #2), except that there, the investment good sector is completely labor intensive. Also, that model is incapable of generating growth. New investment goods contribute to an increase in the stock of capital as follows:

$$k_{t+1} = (1-\delta)k_t + I_t. \tag{8}$$

Suppose the parameter values satisfy the following restrictions:

$$\beta(1 - \delta + b) > 1; \tag{9}$$

and

$$\beta(1 - \delta + b)^{\alpha(1-\sigma)} < 1. \tag{10}$$

Parameter values like $\beta = 1/1.03$; $\delta = 0.10$; $b = 0.14$; $\alpha = 0.36$; $\sigma = 1.5$ satisfy these restrictions. (Think of the model time period as corresponding to one year.) The first condition, (??), guarantees that the efficient allocations, and the equilibrium allocations, display growth, and the second condition guarantees the `finiteness' condition discussed in the previous question. These observations will (hopefully!) become obvious as you work this problem.

(a) Write this problem in the following sequence form:

$$\max_{k_{t+1}, l_t \in B(k_t)} \sum_{t=0}^{\infty} \beta^t F(k_t; k_{t+1}; l_t).$$

Specify what the function F and the constraint set B look like. Be clear on the domain of each of these mappings. (Hint: mimic the steps you took with the two sector model studied in homework #6, question 2.)

(b) Explain why this problem does not satisfy the boundedness restrictions used in Chapter 4 of S-L. (Hint: what would happen to the magnitude of the capital stock if all resources were put into investment?)

(c) The strategy you used to get around this type of problem in homework #6, question 3, applies almost exactly here. Show that the analog of the result established in question 3 (a) holds here, including the boundedness (though perhaps not the sign) feature of v. (Hint: scale c_t by k_t^{α}.) Let the choice variables in this problem (i.e., the ones that correspond to the single choice variable θ_t there) be

$$\lambda_t = \frac{k_{t+1}}{k_t} \text{ and } l_t. \tag{11}$$

Write the analog of the infinite sum in 3(a) separately for the cases $\sigma = 1$ and $\sigma \neq 1$.

(d) Establish the analog of the functional equation in 3(b) of homework #6. Be explicit about the constraint sets on the choice variables.

(e) Establish the analog of the result in 3(c). Do you have the $c = 0$ problem mentioned there when $0 < \sigma < 1$?

(f) Argue that constant values for λ and l solve the sequence problem, and that these values are:

$$\lambda = [\beta(1 - \delta + b)]^{\frac{1}{1-\alpha(1-\sigma)}}, \quad \frac{\eta l}{1 - l} = 1 - \alpha. \tag{12}$$

(Note: $1 - \alpha(1 - \sigma) > 0$ because $\alpha < 1$ and $(1 - \sigma) < 1$.) To the extent that you can, give some intuition about the relationship of the optimal value of λ to the parameters, $\alpha, \beta, \delta, b, \sigma$. What about l and η, α?

3. The point of this exercise is to help familiarize the student with the idea of a sequence-of-markets equilibrium, and to show that this form of market organization constitutes an efficient way to decentralize economic activity in the model economy in the previous question.

Let

$p_t \sim$ date t price of investment goods
$r_t \sim$ date t rental rate on capital
$w_t \sim$ date t wage rate
$\pi_t^i \sim$ date t average firm profits in the i^{th} sector, $i = c, I$.

All these prices are denominated in units of the date t consumption good. Thus, if the consumption good is peanuts, then r_t denotes the number of peanuts that have to be paid to rent one unit of capital for one period.

In a sequence of markets equilibrium, agents meet in markets at each date to trade goods for that date only. At date t, the typical household offers its entire holding of capital, k_t, into the capital rental market. It has no reason to hold any of it back, as long as $r_t > 0$ and renting it out does not result in any wear and tear. (In the model, the stock of capital depreciates at the rate δ, regardless of whether it is used -

fancier models make depreciation an increasing function of usage.) In addition, the household goes to market with a demand for consumption, c_t, and investment, I_t, and a supply of labor l_t. Because of the dynamic nature of its budget constraint and preferences, for the household to select its date t variables, it actually has to form a plan about what it will do forever into the future. Thus, to decide at date t on c_t, I_t, l_t, the typical household must solve:

$$\max_{\{c_r, l_r, k_{r+1}, I_r; r \geq t\}} \sum_{r=t}^{\infty} \beta^{r-t} u(c_r, l_r),$$

subject to:

$$c_r + p_r I_r \leq w_r l_r + r_r k_r + \pi_t^c + \pi_t^I, \; k_{r+1} \leq (1-\delta)k_r + I_r, \; \text{all } r \geq t,$$

the given value of k_t, and the non-negativity constraints. That the typical household receives average profits of all firms reflects that it is perfectly diversified across firms. After doing this at date t, the household moves forward in time to $t+1$, and repeats the process, now treating k_{t+1} as a state variable. It goes on like this, period after period, for an eternity.

At date t, the typical firm that produces consumption goods maximizes profits:

$$\pi_t^c = \max_{c_t, l_t, k_{ct}} (c_t - w_t l_t - r_t k_{ct}),$$

where l_t denotes how much labor the firm hires, k_{ct} denotes how much capital it hires, and c_t denotes the amount of consumption goods it produces. It optimizes subject to the technology, (??), and the non-negativity constraints on inputs and outputs. The firm takes the wage rate and rental rate on capital as given.

At date t, the typical investment good producing firm maximizes its profits:

$$\pi_t^I = \max_{I_t, k_{It}} (p_t I_t - r_t k_{It}),$$

where k_{It} denotes the capital rented by the typical investment good firm, and I_t denotes its output, which is determined by (??). Note that I_t has to be multiplied by p_t to convert I_t into consumption units.

Note that both types of firms pays the same rental rate on capital. That reflects that they both go to a single market, the one where households are supplying k_t; to rent capital.

Definition 1 A sequence-of-markets competitive equilibrium is a set of prices $\{p_t; r_t; w_r; t \geq 0\}$ and a set of quantities, $\{c_t; I_t; l_t; k_{ct}; k_{It}; k_t; t \geq 0\}$ such that:

- Given the prices, the quantities solve the household problem at every date, t:
- Given the prices, the quantities solve the firm problems at every date t:
- Resource constraints, (??)-(??), are satisfied.

(a) Write out the Euler equations and transversality condition of the household, and the first order conditions of the firms. There cannot be an equilibrium with the property $p_t b \neq r_t$ at some t: Explain why not. Explain why profits must be zero in an equilibrium.

(b) Substitute out for the prices in the household Euler equations using the firms' first order conditions. Show that the equilibrium growth rate of capital in the consumption sector, k_{ct+1}/k_{ct}; and employment, l_t; are constants for all time, and correspond to their efficient levels, (??). The employment result is trivial to verify, but the other result is (slightly) more difficult. But, what about k_{c0} and k_{t+1}/k_t? The mere observation that k_{ct} grows at some particular rate does not pin down the growth rate of k_t: Are the values of $k_{t+1}/k_t; t \geq 0$ pinned down by the Euler equations, the transversality condition? Explain.

(c) Consider the parameter values cited above. What is the magnitude of the rate of return on investment? What is $100(p_{t+1}/p_t - 1)$; i.e., the percentage rate at which the price of investment goods is falling? What is the percent rate of growth in consumption, $100(c_{t+1}/c_t - 1)$? What is the percent rate of growth in the stock of capital used in the consumption sector? Show that when the stock of capital in the consumption sector is valued in consumption units (i.e., multiplied by p_t); then the ratio of capital valued in this way, to consumption, is constant in the model. This model is qualitatively consistent with the US data, which indicates a secular fall in the price of investment goods, a rise in the ratio of capital goods to consumption, and no trend in the ratio of capital goods to consumption goods when both are measured in the same units. A shortcoming of this model is that there is little empirical evidence that the capital sector is hugely more capital intensive than the consumption goods sector, as the model requires.

4. Following is another type of endogenous growth model. In this model growth is a consequence of significant complementarities between the activities of one firm and those of other firms. (For further discussion of related ideas, see Romer, Journal of Political Economy, October, 1986, and American Economic Review, May 1987, pp. 56-62.) When the complementarities are great enough, the type of exotic phenomena associated with complementarities that I described in the first lecture begin to emerge. I'll show that in class.

   Following is a description of the economy. The typical household (there are lots of them, all identical) choose consumption, $\tilde{c}_t$, hours worked, $\tilde{n}_t$, and gross investment, $\tilde{k}_{t+1} - (1-\delta)\tilde{k}_t$, to maximize $\sum_{j=0}^{\infty} \beta^t u(\tilde{c}_t, \tilde{n}_t)$ subject to their budget constraint,

$$\tilde{c}_t + \tilde{k}_{t+1} - (1-\delta)\tilde{k}_t = r_t \tilde{k}_t + w_t \tilde{n}_t + \pi_t,$$

   for $t = 1, \ldots, \infty$. They take the initial stock of capital, $\tilde{k}_0$, as given, as well as market prices and profits, $r_t, w_t, \pi_t, t = 1, \ldots, \infty$. The utility function is:

$$u(c, n) = \log(c) + \sigma \log(1-n).$$

   The typical firm hires labor and capital and uses these to produce output, $\tilde{y}_t$, using the production technology,

$$\tilde{y}_t = A_t F(\tilde{k}_t, \tilde{n}_t)$$

   where

$$F(\tilde{k}_t, \tilde{n}_t) = \tilde{k}_t^{\theta} \tilde{n}_t^{(1-\theta)},$$

   and

$$A_t = y_t^{\gamma}, \gamma \geq 0,$$

y_t denotes average, economy-wide output. The typical firm chooses $\tilde{k}_t$ and $\tilde{n}_t$ to maximize profits:

$$\pi_t = \tilde{y}_t - r_t \tilde{k}_t - w_t \tilde{n}_t,$$

treating prices and A_t as given and beyond its control. The object, A_t, is an externality. It captures the notion that a high degree of activity in the economy generally might shift individual firms' production functions up. This may reflect that when there is a high amount of activity, new, creative ideas are being generated more rapidly, and these are freely transferable across firms. If this effect is strong enough, it may offset the depressing effect of investment on the marginal product of capital and permit sustained growth.

A sequence of markets competitive equilibrium is a set of prices and profits, $\{r_t, w_t, \pi_t\}$, and allocations, $\{\tilde{c}_t, \tilde{k}_t, \tilde{n}_t\}$, such that the typical household and firm maximizes (with perfect foresight about later prices) and markets clear. For the goods market, market clearing means that the aggregate resource constraint must be satisfied:

$$c_t + k_{t+1} - (1 - \delta)k_t \leq A_t F(k_t, n_t),$$

for all t, where a variable without a tilde, c, denotes its average, economy-wide value. Since everyone is identical, it seems natural to consider only equilibria in which everyone does the same thing, i.e., tilde'd variables equal their un-tilde'd counterparts.

(a) What is the value of profits, π_t, in equilibrium? Explain.

(b) Write out the first order conditions that the allocations in competitive equilibrium must satisfy. Write them in such a way that prices and profits have been substituted out. Write them in terms of aggregate, economy-wide variables.

(c) Write out the planning problem for this economy in sequence form. Write the first order conditions for this problem, which are analogous to the objects you derived for the competitive equilibrium above. Any interesting differences?

(d) Define a steady-state balanced growth path as a situation in which

$$c_{t+1} = \lambda c_t;\ k_{t+1} = \lambda k_t;\ n_t = n;$$

where λ is the gross growth rate of the economy.

(e) Suppose $\theta/(1-\gamma) = 1$. Show that $\lambda > 1$ is possible.