Christiano
D11-2, Winter 1996

ANSWERS TO MIDTERM EXAM

1. By the fixed point properties of v and $\hat{v}$, $\rho(v,\hat{v}) = \rho(Tv, T\hat{v}) \leq \beta\rho(v,\hat{v})$, where the inequality makes use of the fact that T is a contraction. But, this can only be satisfied by $\rho(v,\hat{v}) = 0$. This in turn is what we mean by v and $\hat{v}$ being 'the same'.

2. My answer to this question is much longer than you could feasibly do in an exam situation. I will use two results to answer the question: the Theorem of the Maximum, and the Benveniste and Scheinkman theorem. The proof style will mimick that of Theorem 4.11 in the book. That theorem does not apply literally, since it is applied in a different environment. Still, the basic strategy works.

   The Benveniste and Scheinkman theorem states: Suppose $X \subseteq \Re^k$ convex, $V : X \to \Re$ concave, $x_0 \in int(X)$, and D is a neighborhood of x_0. Also, W is (i) concave and differentiable on D; (ii) $W(x_0) = V(x_0)$; (iii) $W(x) \leq V(x)$, $x \in D$. Then, V is differentiable at x_0 with $V'(x_0) = W'(x_0)$.

   It is convenient to exploit the strictly increasing property of the utility function so that $c = w(1-l) + rk$. Substitute this into the utility function, and the problem becomes

   $$v(k) = \max_{0 \leq l \leq 1} u\left(w(1-l) + rk, l\right),$$

   where I dropped r and w from v for convenience. The existence of v is guaranteed by the continuity of u and by the fact that the constraint set is non-empty and compact. To establish concavity of v, consider $k_1, k_2, k_\theta > 0$, where $k_\theta = \theta k_1 + (1-\theta)k_2$ for $\theta \in (0,1)$. Also, let l_1, l_2 be the leisure decision when $k = k_1, k_2$, and let $l_\theta = \theta l_1 + (1-\theta)l_2$. (Note, l_θ is not necessarily the leisure decision taken when $k = k_\theta$.) Then,

   $$\begin{aligned} & \theta v(k_1) + (1-\theta)v(k_2) \\ = \; & \theta u(w(1-l_1) + rk_1, l_1) + (1-\theta)u(w(1-l_2) + rk_2, l_2) \text{ (by def. of } l_i, k_i, i = 1, 2) \\ \leq \; & u(w(1-l_\theta) + rk_\theta, l_\theta) \text{ (by concavity of } u) \\ \leq \; & v(k_\theta) \text{ (by the fact that } v(k_\theta) \text{ is the maximized),} \end{aligned}$$

so that concavity is established.

The remainder of the proof mimicks the strategy in the proof of Theorem 4.11, which applies the Benveniste-Scheinkman theorem to a problem different from the present one. Let l^0 be the chosen value of l when $k = k^0$, for some k^0. Then, there is a neighborhood, D, about k^0 such that l^0 is feasible for all $k \in D$. Define $W : D \to \mathbb{R}$:

$$W(k) = u\left(w(1 - l^0) + rk, l^0\right) \leq v(k).$$

The weak inequality holds for all $k \in D$, with equality for $k = k^0$. The conditions of the B-S theorem are satisfied. As a result, the derivative of v exists at $k = k^0$, and that derivative is $u_c(w(1 - l^0) + rk^0, l^0)\, r$.

3. Answer to Arrow-Debreu question.

   (a) Let $p(s^t)$ be the price of a unit of output delivered in history s^t, for all s^t. Let $w(s^t)$ be the price, denominated in units of the s^t consumption good, in s^t, and let $r(s^t)$ denote the real rental rate on capital in s^t, denominated in units of the period s^t consumption. Given prices for all dates and states, $\{p(s^t), r(s^t), w(s^t)$, all $t \geq 0$, all $s^t\}$ the typical household's problem is to choose quantities to maximize present discounted expected utility subject to its budget constraint:

$$\sum_{t=0}^{\infty} \sum_{s^t} p(s^t) \left[c(s^t) + k(s^t) - (1 - \delta)k(s^{t-1})\right]$$
$$\leq \sum_{t=0}^{\infty} \sum_{s^t} p(s^t) \left[r(s^t)k(s^{t-1}) + w(s^t)n(s^t)\right] + \pi,$$

   where π denotes profits.

   The typical firm takes prices as given solves:

$$\pi = \max_{\{y(s^t), k(s^{t-1}), n(s^t);\ \text{all } s^t\}} \sum_{t=0}^{\infty} \sum_{s^t} p(s^t) \left[y(s^t) - r(s^t)k(s^{t-1}) - w(s^t)n(s^t)\right].$$

   Then,

Definition 1 An Arrow-Debreu equilibrium is a level of profits, π; a set of prices, $\{p(s^t); r(s^t); w(s^t);$ all $t \geq 0$; all $s^t\}$; and a set of quantities, $\{y(s^t); c(s^t); k(s^t); n(s^t)$; all $s^t\}$ satisfying the following characteristics:

(i) given the profits and prices, the quantities solve the household problem

(ii) given the prices, the quantities and profits solve the firm problem

(iii) the goods market clears: $y(s^t) = c(s^t) + k(s^t) - (1 - \delta)k(s^{t-1})$; for all s^t:

Given the way I stated (i) and (ii), there was no need to add in (iii) that labor and capital rental markets clear, because the definition specifies that the same $n(s^t)$ and $k(s^{t-1})$ solve the firm and household problems.

Although employment and other variables move up and down (as discussed in the first lecture), this economy nevertheless is efficient. That is, the quantities in competitive equilibrium coincide with the ones that a benevolent planner would choose.

4. Answer:

   (a) $F(k; k') = u\left(k^{\theta}(1 - k')^{(1-\theta)}\right)$ and $\Gamma(k) = K$:

   (b) Result (i) follows from convexity of K; that Γ is non-empty, compact-valued and continuous: $F : A \to \Re$; $A = K \times K$ is bounded and continuous, and $0 < \beta < 1$: Boundedness of F follows from its continuity, and the fact that its domain is bounded. Result (ii) follows from the assumptions in (i) and the fact that F is strictly increasing in its first argument and that Γ is monotone. Result (iii) follows from the assumptions in (i), plus strict concavity of F and convexity of Γ: Result (iv) follows from the assumptions in (i) and (iii), plus differentiability of F: The single-valuedness and continuity of g follows from the Theorem of the maximum, which requires strict concavity of F and v and convexity of Γ:

   (c) When $k = 0$; the marginal product of labor in the production of consumption goods is zero, so all available labor will be allocated to the production of capital goods.

(d) Given the strict concavity assumed for F and result (iv) for v, $F(k,k') + \beta v(k')$ is a strictly concave function in k'. In addition, because of the differentiability assumed for F, result (iv) for v, and the convexity of the constraint set, $\Gamma(k)$, an interior maximum at $k' = g(k)$ implies that the derivative of $F + \beta v$ with respect to k' is zero at $k' = g(k)$.

(e) Note that the 'marginal cost of investment' (MC) is:

$$-F_2(k,k') = (1-\theta)\frac{k^{\sigma\theta}}{(1-k')^{[1-(1-\theta)\sigma]}}.$$

Evidently, MC is strictly increasing in k', and as $k' \to 1$, it goes to $+\infty$. As $k' \to 0$, on the other hand, this expression goes to a well-defined positive number. At the same time, strict concavity of v implies that the 'marginal benefit' (MB) of investment, $\beta v'(k')$, is strictly decreasing in k'. For interior k, the intersection of MB and MC occurs at a value of k' strictly inside the unit interval. What happens to investment with a change in k depends on the impact on the MC and MB curves. The MB curve is invarient to k (from the perspective of tomorrow, today's beginning-of-period capital stock is irrelevant history) so only the MC curve is affected by a change in k. But note from above, that curve shifts up, i.e., that $F_{12} < 0$. Because it shifts up, it follows that $g(k)$ is decreasing in k.

The intuition for this result centers on the MC of investment. The marginal utility cost of extra investment is the product of (i) the marginal loss in consumption resulting from the having to shift a worker out of the consumption sector and into the investment good sector, times (ii) the marginal utility of consumption. The impact of more k for each level of k' on MC depends on the impact on these two terms. In the model, these two terms respond in opposite directions as k is increased for fixed k'. The first term goes up, because the productivity of workers in the consumption goods sector is an increasing function of k, while productivity is invarient to k in the investment goods sector. The second term falls with more k, for fixed k', because of concavity in the utility function. When $\sigma > 0$, then the utility function is not very concave, and so

(i) wins. Evidently, as curvature is increased, with ¾ < 0; then concavity in the utility function wins, in which case $F_{12} > 0$ and g is an increasing function. In the standard growth model, factor (i) is entirely absent. The amount of consumption goods you give up by increasing k^0 by one unit is always one, regardless of the value of k: It is because only factor (ii) is present that the standard model robustly implies $F_{12} > 0$ and that g is increasing.