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D11-2, Winter 1996

MIDTERM EXAM

There are five questions. The total number of possible points is 100, and the number of points per question is given in parentheses. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have 2 hours. Good luck!

1. (5) Suppose $T : S \to S$ is a contraction mapping on a metric space, (S, ρ). Suppose $v, \hat{v} \in S$ and $Tv = v$, $T\hat{v} = \hat{v}$. Show that $v = \hat{v}$.

2. (15) Consider a household which solves the following problem:

$$v(k, r, w) = \max_{c, l \in B(k, r, w)} u(c, l),$$

where $u : \Re_+^2 \to \Re$ is a strictly concave, twice continuously differentiable, strictly increasing function in its two arguments: consumption, c; and leisure, l. The constraints the household must obey in selecting c, l are summarized by B:

$$B(k, r, w) = \{c, l : 0 \le c \le rk + w(1 - l);\ 0 \le l \le 1\}.$$

Here, $r > 0$ is the market rental rate on capital and $w > 0$ is the market wage rate, neither of which the household can control. Also, $k > 0$ is the household's stock of capital. Prove that the derivative of v with respect to k exists, and display a formula for it. If you make use of a theorem to help prove your result, be sure to state it clearly.

3. (20) Suppose that at each date, t, a random variable, s_t, is drawn from a finite set of possible values. Let the history of realizations of s_t be denoted by $s^t = (s_0, s_1, \ldots, s_t)$. The probability of history s^t is $\mu(s^t)$. Let $c(s^t)$ and $n(s^t)$ denote the levels of consumption and labor in history s^t. The stock of capital at the end of period t in history s^t is denoted by $k(s^t)$. The beginning of period t capital, in history s^t is $k(s^{t-1})$. The economy is populated by a large number of identical households, each

of which has preferences over consumption and labor in all states and dates as follows:

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) u(c(s^t), n(s^t)),$$

where $u : \Re^2_+ \to \Re$ is strictly concave, strictly increasing in its first argument and strictly decreasing in its second argument (labor). The technology for converting capital and labor services into output is as follows:

$$c(s^t) + k(s^t) - (1 - \delta)k(s^{t-1}) \leq F(k(s^{t-1}), n(s^t), s_t),$$

where F is linear homogeneous, strictly increasing and strictly concave in each of its first two arguments. The third argument shows how the randomness actually enters this economy: it shifts around the amount of output that results from a given quantity of inputs.

(a) Assume that households own all factors of production, and that they are all endowed with shares in the firms. The initial capital and ownership shares on firms are distributed equally across all households. Describe a market decentralization of this economy in which everyone meets exactly once at date 0 to make their transactions. After the market closes in period 0, households in each subsequent date and state passively deliver the quantities of factors and goods they contracted to sell and receive in the date 0 market. Provide a formal statement of this Arrow-Debreu equilibrium. (Hint: you will have to set up the household and firm maximization problems and specify prices for all the commodities.)

(b) Along a given history, employment in an A-D equilibrium will move up and down as the exogenous shock randomly bumps the production function around. Is the economy inefficient in any of the states of the world, for example the ones in which employment is low? You may provide an informal answer to this question.

4. (40) Consider the following two-sector economy. Consumption goods are produced using the following technology:

$$c_t = k_t^{\theta} n_{1t}^{(1-\theta)}, \quad 0 < \theta < 1.$$

Capital goods depreciate completely in one period and are produced using labor only:

$$k_{t+1} = n_{2t}.$$

The total amount of labor in this economy is 1, so the following condition must be satisfied:

$$n_{1t} + n_{2t} = 1.$$

Let preferences be given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $u : \Re_+ \to \Re$ is strictly concave, strictly increasing and twice continuously differentiable.

(a) (2) Consider the following recursive representation (FE) of the above sequence problem (SP):

$$v(k) = \max_{k' \in \Gamma(k)} F(k, k') + \beta v(k'), \text{ for all } k \in K \equiv \{k : 0 \leq k \leq 1\},$$

with

$$g(k) = \arg\max_{k' \in \Gamma(k)} F(k, k') + \beta v(k'), \text{ for all } k \in K.$$

Give an expression for F and Γ.

(b) (10) It is known that $v : K \to \Re$ in the above expression (i) exists and is unique; (ii) is strictly increasing; (iii) is strictly concave; and (iv) is differentiable for 'interior k', i.e., $k \in \operatorname{int}(K)$ with $g(k) \in \operatorname{int}(\Gamma(k))$. Also, g is single valued and continuous. Explain which particular features of (K, Γ, F, β) guarantee each of these results. It is sufficient to just state whatever theorems you appeal to.

(c) (5) Show that $g(0) = 1$.

(d) (8) Show that, for all interior k, $g(k)$ satisfies

$$F_2(k, g(k)) + \beta v'(g(k)) = 0.$$

(e) (15) Assume $u(c) = c^{\sigma}/\sigma$, where $0 < \sigma < 1$. Show formally that $g(k)$ falls as k increases. In the simple growth model discussed in class (i.e., the one in which $c + k' \leq k^{\theta} + (1 - \delta)k$), $g(k)$ increases as k increases. Give the mathematical reason for the difference in results. Explain the economic intuition underlying the difference.

5. (20) Let $X \subseteq \Re^{l}$; let the correspondence $\Gamma : X \to X$ be nonempty; let $A = \{(x, y) \in X \times X : y \in \Gamma(x)\}$; let $F : A \to \Re$ be a bounded, continuous function; and let $0 < \beta < 1$. Let $B(X)$ be the set of bounded continuous functions $f : X \to \Re$, with the sup norm. Let H be the set of continuous functions $h : X \to X$ such that $h(x) \in \Gamma(x)$, all $x \in X$. For any $h \in H$, define the operator T_h on $B(X)$ by $(T_h f)(x) = F[x, h(x)] + \beta f[h(x)]$.

   Argue that T_h has a unique fixed point $w \in B(X)$. (You may have to appeal to a theorem - if so, just state it as best as you can.)