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D11-2, Winter 1997
Homework 2. Due: Wednesday, January 22.

1. Consider the following two-sector economy. Consumption goods are produced using the following technology:

$$c_t = k_t^{\theta} n_{1t}^{(1-\theta)}, \; 0 < \theta < 1.$$

Capital goods depreciate completely in one period and are produced using labor only:

$$k_{t+1} = n_{2t}.$$

The total amount of labor in this economy is 1, so the following condition must be satisfied:

$$n_{1t} + n_{2t} = 1.$$

Let preferences be given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly concave, strictly increasing and twice continuously differentiable.

(a) Consider the functional equation (FE) associated with the above sequence problem (SP):

$$v(k) = \max_{k' \in \Gamma(k)} F(k, k') + \beta v(k'), \text{ for all } k \in K \equiv \{k : 0 \le k \le 1\},$$

with

$$g(k) = \arg\max_{k' \in \Gamma(k)} F(k, k') + \beta v(k'), \text{ for all } k \in K.$$

Give an expression for F and Γ.

(b) It is known that $v : K \to \mathbb{R}$ in the above expression (i) exists and is unique; (ii) is strictly increasing; (iii) is strictly concave; and (iv) is differentiable for 'interior k', i.e., $k \in \text{int}(K)$ with $g(k) \in \text{int}(\Gamma(k))$. Also, g is single valued and continuous. Explain which particular features of (K, Γ, F, β) guarantee each of these results. It is sufficient to just state whatever theorems you appeal to.

(c) Explain why $g(0) = 1$.

(d) Show that, for all interior k, $g(k)$ satisfies

$$F_2(k, g(k)) + \beta v'(g(k)) = 0.$$

(e) Assume $u(c) = c^{\sigma}/\sigma$, where $0 < \sigma < 1$. Show that $g(k)$ falls as k increases. In the simple growth model discussed in class (i.e., the one in which $c + k' \leq k^{\theta} + (1 - \delta)k$), $g(k)$ increases as k increases. Give the mathematical reason for the difference in results. Explain the economic intuition underlying the difference.

2. This question asks you to redo Theorem 4.15 in a model that incorporates hours worked as a choice variable. Consider the following utility function:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t), \qquad (1)$$

where $c_t \geq 0$ and $0 \leq n_t \leq 1$ denote date t consumption and employment, respectively. The resource constraint is:

$$c_t + k_{t+1} \leq f(k_t, n_t), \qquad (2)$$

with $k_t \geq 0$. Here, u is strictly concave, differentiable, strictly increasing in c and strictly decreasing in n. Also, f is strictly increasing in each argument, is linearly homogeneous of degree zero, differentiable and concave. Suppose $c_t^*, k_t^* > 0$; $0 < n_t^* < 1$ satisfy (2) at each t and $k_0^* = \bar{k}_0$, the given initial stock of capital. Also, these numbers satisfy the 'Euler equations':

$$\begin{aligned} u_c(c_t^*, n_t^*) &= \beta u_c(c_{t+1}^*, n_{t+1}^*) f_k(k_{t+1}^*, n_{t+1}^*), \\ u_c(c_t^*, n_t^*) f_n(k_t^*, n_t^*) + u_n(c_t^*, n_t^*) &= 0, \end{aligned}$$

for $t = 0, 1, 2, \ldots$, and the 'transversality condition':

$$\lim_{T \to \infty} u_c(c_T^*, n_T^*) k_{T+1}^* = 0.$$

Here, u_c and u_n denote the derivatives of u with respect to its first and second argument, and similarly for f. Show that the given sequences $\{c_t^*, k_t^*, n_t^*; t \geq 0\}$ produce the highest value of (1) within the set of sequences which satisfy (2) and the inequality constraints on consumption, labor and capital. (Hint: imitate the proof to Theorem 4.15 in the text.)

3. Consider the standard one-sector growth model with preferences $\mathbf{P}^{1}_{t=0} 0{:}95^t \log(c_t)$ and resource constraint $c_t + k_{t+1} = k_t^{0:36} + (1 ¡ 0{:}02)k_t$:

    (a) Report the ¯rst and second order Taylor series expansions of the policy rule, g; about the steady state value of k_t:

    (b) Report the amount of periods it would take to close 95% of the gap between actual and steady-state capital if capital falls 20 and 80 percent below its steady state value, respectively. Do these two calculations for each of the two approximations to the policy function, g: Does the second-order term help, or is the linear approximation g adequate.