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D11-2, Winter 1997
Homework 3. Due: Wednesday, January 29.
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1. Suppose a planner chooses to maximize, by choice of $c_0, c_1, c_2, \ldots,$ the following expression:

$$u(c_0) + \delta[\beta u(c_1) + \beta^2 u(c_2) + \ldots], u(c_t) = \log(c_t)$$

subject to

$$c_t = k_t^\alpha - k_{t+1}, 0 < \alpha < 1, c_t, k_{t+1} \geq 0, \; k_0 \text{ given},$$

where $0 < \delta < \beta < 1$. When $\delta = 1$, this is the problem studied in exercises 2.2 and 4.9 in SL.

   (a) Use the fact that $k' = \beta\alpha k^\alpha$ solves the version of the problem with $\delta = 1$ to establish that the solution to the problem with $\delta \neq 1$ has the form:

   $$k_1 = g k_0^\alpha, k_{t+1} = \beta\alpha k_t^\alpha, t = 1, 2, \ldots,$$

   where g is a scalar. Derive an explicit formula relating g to the parameters of the model, β, α, δ.

   (b) Is there a unique k^* with the property $k_t \to k^*$ as $t \to \infty$ for all k_0? Display a formula relating k^* to the parameters of the model.

   (c) Suppose $\beta = 1/1.03, \alpha = .36, \delta = .8$. Suppose $k_0 = k^*$. Display the values of $k_0, k_1, k_2, k_3, k_4, k_5$ that solve the problem as of date zero.

   (d) Now suppose that when date 1 happens, the planner decides to reoptimize with respect to $k_2, k_3, \ldots$. The initial condition for this problem is k_1, the decision implemented by the planner last period. The planner's preferences over $c_t, t \geq 1$ are as follows:

   $$u(c_1) + \delta[\beta u(c_2) + \beta^2 u(c_3) + \ldots]$$

   and the resource constraint is as before. What values will the planner choose for k_1, k_2, k_3, k_4, k_5? If the planner chooses to re-optimize in this way every period, to what value will k_t actually tend?

(e) Are the values for $k_2; k_3; k_4; k_5$ chosen by the planner in date 1 the same as the values for these variables chosen in date 0? Why not? Because the chosen values for these variables differs between time 0 and time 1, this problem is said to be time inconsistent. If δ had been set to one, we would not have had this problem. Why not?

(f) Basically, the attitude of the planner is `I'm very impatient today (the discount rate from period 0 to period 1 is $\beta\delta$), but I'll be less impatient tomorrow (the discount rate from period 1 to period 2 is β), so I'll consume a lot today and save a lot tomorrow.' Such an attitude is not time consistent because when tomorrow rolls around the planner says the same thing. In the end, the planner just ends up with a low capital stock. This type of model has been used to explain the behavior of smokers, who resolve that `tomorrow I'll quit smoking, but tonight I'll just have one or two more'. Does the solution concept that we have used make any sense? Would a rational person make decisions in the time-inconsistent way described in (d) and (e), or would they do something else? Answers to this question often involve posing the problem as a game between the planner in period t and the planner in period t+1, and takes us beyond the scope of this course.

(g) Explain why there is no problem defining an Arrow-Debreu equilibrium when agents have this type of preferences. Does there exist a Sequence of Markets equilibrium?

2. Consider an economy with a large number of identical households, each having preferences, $\sum_{t=0}^{\infty} \beta^t u(c_t)$: Suppose the resource constraint is $c_t + i_t \leq f(k_t)$; where $k_{t+1} = i_t + (1 - \delta)k_t$; f is strictly increasing and concave, $f'(k) \to \infty$ as $k \to 0$; $f'(k) \to 0$ as $k \to \infty$, $0 < \delta < 1$: Assume investment, i_t; is irreversible, i.e., it must be that $i_t \geq 0$: In addition, suppose $c_t; k_t \geq 0$ and that $k_0 > 0$ is given. Consider the functional equation associated with this problem:

$$\begin{aligned} v(k) &= \max_{k' \in \Gamma(k)} u(f(k) + (1 - \delta)k - k') + \beta v(k') \\ \Gamma(k) &= \{k' : (1 - \delta)k \leq k' \leq f(k)\}: \end{aligned}$$

(a) State a set of assumptions on β and u that guarantee there is a unique, differentiable, concave v that solves the above functional equation. For each property of v, explain which assumptions are used to get it.

(b) Show that monotonicity of $\Gamma(k)$, Assumption 4.6 in S-L, fails so that one of the conditions of Theorem 4.7 which guarantee strictly increasing v, is not satisfied.

(c) Show that the feasible set for this economy satisfies the following 'quasi-monotonicity property': if $\bar{k} \geq k$, then $\Gamma(k) + (1-\delta)(\bar{k} - k) \subseteq \Gamma(\bar{k})$. Here, the sum of a set, say X, and a number, say a, is a new set, $X + a$, where $X + a \equiv \{x + a : x \in X\}$.

(d) Show: v is an increasing function in k. (Hint: (i) following the basic strategy of the proof of Theorem 4.7, it's enough to establish that the assumptions of Theorem 4.7 with the monotonicity assumption on Γ replaced by quasi-monotonicity guarantee Tw is increasing if w is; (ii) make use of the fact that if $k' \in \Gamma(k)$, then $\bar{k}' = k' + (1-\delta)(\bar{k} - k) \in \Gamma(\bar{k})$, $\bar{k}' > k'$, and $f(\bar{k}) + (1-\delta)\bar{k} - \bar{k}' > f(k) + (1-\delta)k - k'$.) Can you provide intuition for the fact that v is increasing even though Γ fails to satisfy monotonicity?

3. Consider the economy in question (1), except that $\delta = 1$. Define a recursive competitive equilibrium for this economy, and display explicitly the household's value function ($V(K; k)$ in S-L, p. 30), policy function, $H(K; k)$, the aggregate law of motion for capital, $k' = h(k)$, and the aggregate pricing functions, $R(k)$ and $\omega(k)$, for this economy.