

Homework 4
 Due Wednesday, February 5.
 D11-2, Winter 1997
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1. This question and the next studies an endogenous growth model that I had intended to cover in class, but cannot due to time constraints. The question considers a model which posits that the production of new investment goods is heavily capital-intensive, and does not involve diminishing returns. (See Sergio Rebelo, 'Long-run Policy Analysis and Long-Run Growth,' Journal of Political Economy, June 1991, and Jones and Manuelli, section 4 of their 'The Sources of Growth,' Journal of Economic Dynamics and Control, Vol. 21, no. 1, January 1997.)

The model in this homework generates growth in a way that is also consistent with the empirical observation that the price of new investment goods (particularly business equipment and household durables, but also structures) has been falling rapidly relative to the cost of consumption goods. The model, in effect, takes the position that this observation is the key to understanding the factors underlying growth in the post war period.

The economy has a large number of identical agents, each having preferences $\sum_{t=0}^{\infty} \beta^t u(c_t; I_t)$, with

$$u(c; I) = \begin{cases} [c(1 - \lambda)^\lambda]^{1-\frac{1}{\sigma}} = (1 - \lambda)^{\frac{1-\sigma}{\sigma}}; & \text{for } \sigma > 0; \sigma \leq 1 \\ \log(c) + \lambda \log(1 - \lambda); & \text{for } \sigma = 1: \end{cases} \quad (1)$$

The assumption that u is strictly increasing in c and strictly decreasing in I ; and strictly concave corresponds to:

$$\sigma > 0; \lambda > 0; \lambda(1 - \lambda) < 1; \sigma > \lambda(1 - \lambda):$$

There is no uncertainty in this economy. There are two separate technologies: one for producing the consumption good, c_t ; and one for producing the investment good, I_t : The first technology is:

$$c_t = A k_{ct}^\alpha I_t^{1-\alpha}; \quad (2)$$

where A is a positive constant, k_{ct} is capital used in the consumption sector, and $0 < \alpha < 1$: The technology for producing investment goods is:

$$I_t = b k_{it}; \quad (3)$$

where k_{it} is the stock of capital used in the investment sector and b is a positive constant. At any particular point of time, t ; the aggregate stock of capital, k_t ; is given. There are no restrictions on how that capital may be allocated between the two sectors, subject to:

$$k_t = k_{ct} + k_{it}; \quad k_{ct} \geq 0; \quad k_{it} \geq 0; \quad (4)$$

The model is like the one analyzed in homework 2 (question #2), except that there, the investment good sector is completely labor intensive. Also, that model is incapable of generating growth. New investment goods contribute to an increase in the stock of capital as follows:

$$k_{t+1} = (1 - \delta)k_t + I_t; \quad (5)$$

Suppose the parameter values satisfy the following restrictions:

$$-(1 - \delta + b) > 1; \quad (6)$$

and

$$-(1 - \delta + b)^{\alpha(1 - \beta)} < 1; \quad (7)$$

Parameter values like $\delta = 0.10$; $\alpha = 0.36$; $\beta = 0.14$; $\beta = 1.5$ satisfy these restrictions. (Think of the model time period as corresponding to one year.) The first condition, (6), guarantees that the efficient allocations, and the equilibrium allocations, display growth, and the second condition guarantees finiteness of the planner's objective. These observations will (hopefully!) become obvious as you work this problem.

(a) Write this problem in the following sequence form:

$$\max_{\{k_{t+1}; I_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(k_t; k_{t+1}; I_t):$$

Specify what the function F and the constraint set \mathcal{J} look like. Be clear on the domain of each of these mappings.

- (b) Explain why this problem does not satisfy the boundedness restrictions used in Chapter 4 of S-L.
- (c) The strategy we have used several times in class to get around the boundedness problem applies almost exactly here. (Hint: scale c_t by k_t^α .) Let the choice variables in this problem be

$$s_t = \frac{k_{t+1}}{k_t} \text{ and } l_t \quad (8)$$

- (d) Identify a functional equation that is useful for studying this problem. Be explicit about the constraint sets on the choice variables. Explain why there is only one functional equation for this problem.
- (e) Argue that constant values for s and l solve the sequence problem, and that these values are:

$$s = \left[\frac{1 - \alpha + b}{1 - \alpha} \right]^{\frac{1}{1 - \alpha}} ; \frac{l}{1 - l} = \frac{1}{1 - \alpha} \quad (9)$$

(Note: $1 - \alpha > 0$ because $\alpha < 1$ and $1 - \alpha < 1$.) To the extent that you can, give some intuition about the relationship of the optimal value of s to the parameters, α ; b ; α : What about l and α ?

2. We will now consider a sequence-of-markets decentralization of the economy in the previous question.

Let

- p_t » date t price of investment goods
- r_t » date t rental rate on capital
- w_t » date t wage rate
- π_t^i » date t average firm profits in the i^{th} sector, $i = c; I$.

All these prices are denominated in units of the date t consumption good. Thus, if the consumption good is peanuts, then r_t denotes the number of peanuts that have to be paid to rent one unit of capital for one period.

In a sequence of markets equilibrium, agents meet in markets at each date to trade goods for that date only. At date t ; the typical household

offers its entire holding of capital, k_t , into the capital rental market. It has no reason to hold any of it back, as long as $r_t > 0$ and renting it out does not result in any wear and tear. (In the model, the stock of capital depreciates at the rate δ ; regardless of whether it is used - fancier models make depreciation an increasing function of usage.) In addition, the household goes to market with a demand for consumption, c_t ; and investment, I_t ; and a supply of labor l_t : Because of the dynamic nature of its budget constraint and preferences, for the household to select its date t variables, it actually has to form a plan about what it will do forever into the future. Thus, to decide at date t on c_t, I_t, l_t ; the typical household must solve:

$$\max_{\{c_r, l_r; k_{r+1}; r_s\}_{r=t}} \sum_{r=t}^{\infty} \beta^{r-t} u(c_r, l_r);$$

subject to:

$$c_r + p_r l_r + w_r l_r + r_r k_r + \frac{1}{4}_t^c + \frac{1}{4}_t^I; k_{r+1} = (1 - \delta)k_r + I_r; \text{ all } r \geq t;$$

the given value of k_t ; and the non-negativity constraints. That the typical household receives average profits of all firms reflects that it is perfectly diversified across firms. After doing this at date t ; the household moves forward in time to $t + 1$; and repeats the process, now treating k_{t+1} as a state variable. It goes on like this, period after period, for an eternity.

At date t ; the typical firm that produces consumption goods maximizes profits:

$$\frac{1}{4}_t^c = \max_{c_t, l_t, k_{ct}} (c_t - w_t l_t - r_t k_{ct});$$

where l_t denotes how much labor the firm hires, k_{ct} denotes how much capital it hires, and c_t denotes the amount of consumption goods it produces. It optimizes subject to the technology, (2), and the non-negativity constraints on inputs and outputs. The firm takes the wage rate and rental rate on capital as given.

At date t ; the typical investment good producing firm maximizes its profits:

$$\frac{1}{4}_t^I = \max_{l_t, k_{It}} (p_t I_t - w_t l_t - r_t k_{It});$$

where k_{1t} denotes the capital rented by the typical investment good firm, and l_t denotes its output, which is determined by (3). Note that l_t has to be multiplied by p_t to convert l_t into consumption units.

Note that both types of firms pay the same rental rate on capital. That reflects that they both go to a single market, the one where households are supplying k_t to rent capital.

Definition 1 A sequence-of-markets competitive equilibrium is a set of prices $\{p_t, r_t, w_t, t_{\geq 0}\}$ and a set of quantities, $\{c_t, l_t, k_{ct}, k_{1t}, k_t, t_{\geq 0}\}$ such that:

- 2 Given the prices, the quantities solve the household problem at every date, t :
 - 2 Given the prices, the quantities solve the firm problems at every date t :
 - 2 Resource constraints, (2)-(5), are satisfied.
- (a) Write out the Euler equations and transversality condition of the household, and the first order conditions of the firms. There cannot be an equilibrium with the property $p_t b \neq r_t$ at some t : Explain why not. Explain why profits must be zero in an equilibrium.
- (b) Substitute out for the prices in the household Euler equations using the firms' first order conditions. Show that the equilibrium growth rate of capital in the consumption sector, $k_{ct+1}=k_{ct}$; and employment, l_t ; are constants for all time, and correspond to their efficient levels, (9). The employment result is trivial to verify, but the other result is (slightly) more difficult. But, what about k_{c0} and $k_{t+1}=k_t$? The mere observation that k_{ct} grows at some particular rate does not pin down the growth rate of k_t : Are the values of $k_{t+1}=k_t$; $t \geq 0$ pinned down by the Euler equations, the transversality condition? Explain.
- (c) Consider the parameter values cited above. What is the magnitude of the rate of return on investment? What is $100(p_{t+1}=p_t - 1)$; i.e., the percentage rate at which the price of investment goods is falling? What is the percent rate of growth in consumption,

$100(c_{t+1} = c_t \pm 1)$? What is the percent rate of growth in the stock of capital used in the consumption sector? Show that when the stock of capital in the consumption sector is valued in consumption units (i.e., multiplied by p_t); then the ratio of capital valued in this way, to consumption, is constant in the model. This model is qualitatively consistent with the US data, which indicates a secular fall in the price of investment goods, a rise in the ratio of capital goods to consumption, and no trend in the ratio of capital goods to consumption goods when both are measured in the same units. A shortcoming of this model is that there is little empirical evidence that the capital sector is hugely more capital intensive than the consumption goods sector, as the model requires.

- Following is another type of endogenous growth model. In this model growth is a consequence of significant complementarities between the activities of one firm and those of other firms. (For further discussion of related ideas, see Romer, *Journal of Political Economy*, October, 1986.) Following is a description of the economy. The typical household (there are lots of them, all identical) choose consumption, c_t ; hours worked, n_t ; and gross investment, $I_{t+1} = (1 + \delta)K_t$; to maximize $\sum_{j=0}^{\infty} \beta^j u(c_t, n_t)$ subject to their budget constraint,

$$c_t + I_{t+1} = (1 + \delta)K_t = r_t K_t + w_t n_t + Y_t$$

for $t = 1, \dots, \infty$: They take the initial stock of capital, K_0 ; as given, as well as market prices and profits, $r_t, w_t, Y_t; t = 1, \dots, \infty$: The utility function is:

$$u(c; n) = \log(c) + \frac{1}{2} \log(1 - n)$$

The typical firm hires labor and capital and uses these to produce output, y_t ; using the production technology,

$$y_t = A_t F(K_t, n_t)$$

where

$$F(K_t, n_t) = K_t^\alpha n_t^{1-\alpha}$$

and

$$A_t = y_t^\beta; \beta > 0$$

y_t denotes average, economy-wide output. The typical firm chooses \tilde{r}_t and \tilde{a}_t to maximize profits:

$$\pi_t = y_t - r_t \tilde{r}_t - w_t \tilde{a}_t;$$

treating prices and A_t as given and beyond its control. The object, A_t , is an externality. It captures the notion that a high degree of activity in the economy generally might shift individual firms' production functions up. This may reflect that when there is a high amount of activity, new, creative ideas are being generated more rapidly, and these are freely transferable across firms. If this effect is strong enough, it may offset the depressing effect of investment on the marginal product of capital and permit sustained growth.

A sequence of markets competitive equilibrium is a set of prices and profits, $\tilde{r}_t; w_t; \pi_t$; and allocations, $\tilde{c}_t; \tilde{r}_t; \tilde{a}_t$, such that the typical household and firm maximizes (with perfect foresight about later prices) and markets clear. For the goods market, market clearing means that the aggregate resource constraint must be satisfied:

$$c_t + k_{t+1} - (1 - \delta)k_t = A_t F(k_t; n_t);$$

for all t , where a variable without a tilde, c , denotes its average, economy-wide value. Since everyone is identical, it seems natural to consider only equilibria in which everyone does the same thing, i.e., tilde'd variables equal their un-tilde'd counterparts.

- What is the value of profits, π_t , in equilibrium? Explain.
- Write out the first order conditions that the allocations in competitive equilibrium must satisfy. Write them in such a way that prices and profits have been substituted out. Write them in terms of aggregate, economy-wide variables.
- Write out the planning problem for this economy in sequence form. Write the first order conditions for this problem, which are analogous to the objects you derived for the competitive equilibrium above. Any interesting differences?

(d) Define a steady-state balanced growth path as a situation in which

$$c_{t+1} = \mu c_t; k_{t+1} = \mu k_t; n_t = n;$$

where μ is the gross growth rate of the economy.

(e) Suppose $\mu = (1 - \delta) = 1$: Show that $\mu > 1$ is possible:

(f) Are the allocations in this economy efficient?

4. Show that the allocations in Romer's variety model (the one discussed in class) are not efficient.
5. Identify a decentralization for the human capital economy discussed in the January 29 class.
6. Consider the 'AK' economy discussed in class. Replace the resource constraint in that economy with

$$c + k^\theta + (1 - \delta)k = Ak + Bk^\alpha n^{1-\alpha};$$

Leave the preferences as $\sum_{t=0}^{\infty} \beta^t u(c_t)$:

- (a) Can you construct a competitive framework to support the efficient allocations?
- (b) Does this model display non-trivial transient dynamics? Can it display constant growth? Explain.