Homework #5
Economics D11-2
Due Thursday, February 20
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1. This question asks you to redo Theorem 4.15 in a model that takes into account uncertainty. Suppose that at each date t a random variable, s_t, is realized. It can take on any one of N possible values:

$$s(1), s(2), ..., s(N).$$

Call s_t the state of nature at date t. Let s^t denote the history of states of nature up to time t:

$$s^t = (s_0, s_1, ..., s_t).$$

At date 0, s_0 is known. Thus, as of date 0, there is one possible history, s^0, N possible histories, s^1, N^2 possible histories, s^2, ... N^t possible histories s^t, etc.

Let the probability of history s^t be denoted by $\mu(s^t)$. Then, by the definition of a probability,

$$\mu(s^t) \geq 0, \text{ for all } s^t, \text{ and } \sum_{s^t} \mu(s^t) = 1, \text{ for every } t = 0, 1, 2, ...,$$

where $\sum_{s^t}$ denotes `the sum over all N^t possible values of s^t'.

Let the $N \times N$ matrix π be defined by:

$$\pi_{ij} = \text{Probability}[s_{t+1} = s(j) | s_t = s(i)].$$

(a) Suppose $v(s^t) = v^i$ if $s_t = s(i)$, for $i = 1, ..., N$. That is, the value taken on by $v(s^t)$ is a function only of the current state of nature. Let the $N \times N$ matrix π^2 be defined by $\pi^2 = \pi \times \pi$. Similarly, define $\pi^3 = \pi^2\pi$, ..., $\pi^k = \pi^{k-1}\pi$.

i. Prove that each row of π^k is a probability distribution (i.e., all elements of π^k are non-negative and π^k satisfies $\pi^k\iota = \iota$, where ι is the $N \times 1$ vector $\iota = (1, 1, ..., 1)'$).

ii. Suppose $s_0 = s(k)$: Show that:

$$\sum_{t=0}^{\infty} \sum_{s^t} \mu(s^t)\beta^t v(s^t) = \kappa [I - \beta\pi]^{-1} v; \tag{1}$$

where v is an $N \times 1$ column vector, $v = (v^1; \ldots; v^N)'$; and κ is a $1 \times N$ row vector with all zeros, except a one in the k^{th} entry. Recall the definition of a double sum:

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q_{ij} \equiv [q_{00} + q_{01} + q_{02} + \ldots] + [q_{10} + q_{11} + q_{12} + \ldots] + \ldots$$

(Hint: start by writing the expression on the left of the equality in (??) explicitly for t=0,1,2,..., and stare.)

iii. Show that:

$$\sum_{s^{t+1}} q(s^{t+1}) = \sum_{s^t} \sum_{s^{t+1}|s^t} q(s^{t+1}); \tag{2}$$

where $s^{t+1} | s^t$ signifies 'all possible histories s^{t+1}, given history s^t has occurred'. Also, $q(s^{t+1})$ is an arbitrary variable, indexed by histories. It's enough to establish the result for $t = 1$ and $N = 2$:

(b) Consider the utility function:

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \mu(s^t) u(c(s^t)); \tag{3}$$

and resource constraint:

$$c(s^t) + k(s^t) \leq f(k(s^{t-1}); s_t): \tag{4}$$

Note that s_t shifts the production function. Assume u and f satisfy the conditions in the handout on the canonical model (with the obvious modifications to reflect the absence of hours worked from the problem!).

Suppose $c^*(s^t); k^*(s^t) > 0$ satisfy (??) for all s^t; $t = 0; 1; 2; \ldots$; with $k^*(s^{-1}) = k_0$; the given initial stock of capital. Suppose also that the 'Euler equations' are satisfied:

$$u_c(c^*(s^t)) = \beta \sum_{s^{t+1}|s^t} \frac{\mu(s^{t+1})}{\mu(s^t)} u_c(c^*(s^{t+1})) f_k(k^*(s^t); s_{t+1});$$

for all s^t; $t \geq 0$; and the `transversality condition':

$$\lim_{T \to \infty} \sum_{s^T} \beta^T \pi(s^T) u_c(c^*(s^T)) k^*(s^T) \to 0.$$

Prove that $\{c^*(s^t), k^*(s^t); t \geq 0;$ all $s^t\}$ yield the highest value of (??) within the set of all sequences that satisfy (??) and the nonnegativity constraints on consumption and the stock of capital. (Hint: imitate the proof strategy of Theorem 4.15 as closely as you can, and make use of (??) when you group terms in the capital stock).

The Euler and transversality conditions are sometimes stated using the expectation operator:

$$u_{c,t} = \beta E_t u_{c,t+1} f_{k,t+1}$$

and

$$\lim_{T \to \infty} E_0 \beta^T u_{c,T} k_{T+1} = 0,$$

where E_t denotes the mathematical expectation operator, conditional on information dated t and earlier (to understand the conditional expectation operator in the euler equation, recall that $\frac{\pi(s^{t+1})}{\pi(s^t)}$ signifies the conditional probability of s^{t+1}, given s^t.)

2. Until now we have never worried about how investment is financed. That's because we have always considered decentralizations in which the household buys additions to capital using current income, so there is no need for `finance'. This way of organizing things makes it impossible for us to think about an asset market variable like the rate of return on equity. In this question we consider a decentralization in which the accumulation of capital is financed by entrepeneurs who put the capital to work with hired labor to produce output. The entrepeneurs issue equity and debt to finance their acquisition of capital. As a result, the environment in this question facilitates thinking about the equilibrium rate of return on equity and on other assets such as a one-period-ahead sure loan (our version of corporate debt).

   The environment below has three properties: First, the equilibrium consumption, labor and capital stock quantities in this model are not dependent on the debt-to-equity ratio of the ¯rm. This is a version of the celebrated Modigliani-Miller theorem in ¯nance. Second, the equilibrium rate of return on equity does depend on the debt-to-equity ratio. This is because equity has to absorb all the uncertainty in the ¯rm's revenue stream, and that is riskier as the ¯rm is more leveraged with debt. The premium on the rate of return on equity over debt increases as the debt to equity ratio increases. Third, for this model to account for the empirically observed equity premium requires an implausibly high debt-to-equity ratio (this is consistent with the ¯ndings of a celebrated paper, Mehra and Prescott `The Equity Premium: A Puzzle,' Journal of Monetary Economics, 1985).

Consider the following economy with households and ¯rms.

Households

We suppose there are many identical households. The typical household takes prices, wages and rates of return as given. In a sequence-of-markets competitive environment, the household seeks at time t to maximize expected utility:

$$E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}) \tag{5}$$

subject to the sequence of budget constraints,

$$c_{t+j} + z_{t+j+1} + b_{t+j+1} = w_{t+j} n + R_{t+j} b_{t+j} + r_{t+j} z_{t+j}; \tag{6}$$

$j \geq 0$, and subject to initial levels of the stock of debt and equity, b_t and z_t: Here, c_t denotes consumption and household supply of labor, n; is assumed to be ¯xed. Equity and debt acquired in period t, z_{t+1} and b_{t+1}; have rates of return r_{t+1} and R_{t+1}, respectively. The return, r_{t+1}; is a function of date $t + 1$ (and possibly earlier) economic shocks, while R_{t+1} is only a function of shocks dated t and earlier. Thus, from the perspective of time t; the time $t + 1$ rate of return on debt is constant across date $t + 1$ states of nature, while the rate of return on r_{t+1} varies across date $t + 1$ states of nature. This feature of the return on government debt leads us to refer to is as `conditionally sure'. The household takes $r_t; w_t$ (wages) and R_t as given and beyond its control.

(a) Write out the first order necessary conditions associated with the household's choice of z_{t+1} and b_{t+1}.

(b) Define the date t equity premium, P_t, to be the excess of the date t conditionally expected return on equity, $E_t r_{t+1}$, over the conditional sure return on bonds, R_{t+1}.

$$P_t = \frac{E_t r_{t+1}}{R_{t+1}}.$$

(c) Show that

$$P_t = 1 - \mathrm{Cov}_t(m_{t+1}, r_{t+1}),$$

where $m_{t+1} = \beta u_{c,t+1}/u_{c,t}$, is the intertemporal marginal rate of substitution in consumption and $u_{c,t}$ is the date t marginal utility of consumption. (Hint: use the fact, $\mathrm{Cov}_t(x_{t+1}, y_{t+1}) = E_t x_{t+1} y_{t+1} - E_t x_{t+1} E_t y_{t+1}$, and the first order conditions developed in (a).)

(d) Suppose the conditional covariance in the above expression were positive. Then P_t is less than 1, i.e., the conditionally expected rate of return on equity is less than that on debt. This seems peculiar. Why should people be willing to hold equity when its payoff is uncertain, and lower in expected value than debt? Explain in intuitive terms.

Firms

We suppose there are many firms, all of which are identical. The typical firm takes prices, w_t and rates of return, R_t, r_t, as given. In period $t+1$ the firm uses capital, K_{t+1}, and labor, N_{t+1}, to produce output, Y_{t+1}, using the following production function:

$$Y_{t+1} = F(K_{t+1}, N_{t+1}, \varepsilon_{t+1}), \tag{7}$$

where F is linear homogeneous in its first two arguments and ε_{t+1} is a stationary random shock to technology. An entrepreneur who wishes to operate the firm in period $t+1$ must during period t acquire the purchasing power needed to purchase K_{t+1} from the current firm. The entrepreneur acquires this purchasing power by selling equity shares, Z_{t+1}, and debt, B_{t+1}, to the household. The entrepreneur's period t financing constraint is:

$$Z_{t+1} + B_{t+1} = K_{t+1}. \tag{8}$$

Next period, the entrepreneur brings $Y_{t+1} + (1 - \delta)K_{t+1}$ to the goods market to sell. This includes new production, Y_{t+1}, and the capital stock that remains after depreciation at the end of next period, $(1 - \delta)K_{t+1}$. The entrepeneur's expenses next period include the wage bill, $w_{t+1}N_{t+1}$, and the obligations on debt, $R_{t+1}B_{t+1}$, and equity, $r_{t+1}Z_{t+1}$. Thus, the entrepeneur's total profit at the end of t+1 is π_{t+1}, where,

$$\pi_{t+1} = Y_{t+1} + (1 - \delta)K_{t+1} - w_{t+1}N_{t+1} - R_{t+1}B_{t+1} - r_{t+1}Z_{t+1} \tag{9}$$

The entrepeneur's objective is to maximize π_{t+1}. However, at the time Z_{t+1} and B_{t+1} are chosen, ε_{t+1} is not known. The entrepreneur weighs the different π_{t+1}'s corresponding to different realizations of ε_{t+1} by the product of the probability of that realization and the associated marginal utility of consumption.

(e) Suppose that at date t there were markets for date t + 1 state-contingent goods. Show that the weights that we apply to profits across states of nature correspond to the prices that would obtain in such markets for state-contingent goods.

(f) The entrepreneur in our model is concerned with $E_t u_{c,t+1}\pi_{t+1}$, where $u_{c,t+1}$ is treated as exogenous. The entrepreneur solves

$$\max_{Z_{t+1},B_{t+1}} E_t \{u_{c,t+1} \max_{N_{t+1}} \pi_{t+1}\}. \tag{10}$$

The maximization inside the braces reflects that the firm's employment decision is made after the realization of ε_{t+1}. The maximization outside the braces reflects that the financing decision is made before the realization of ε_{t+1}. Make sure you understand this point.

(g) write out the first order necessary conditions associated with the firm's choice of Z_{t+1}, B_{t+1}, and N_{t+1}.

Suppose that there is free entry, so that, in equilibrium, $E_t\{u_{c,t+1}\pi_{t+1}\} = 0$. Suppose, in addition, that the equilibrium process, r_t, has the property that ex post profits are zero, i.e., $\pi_t = 0$.

(h) Define a sequence of markets equilibrium and a recursive competitive equilibrium for this economy.

(i) Show that the debt-to-equity ratio, b_{t+1}/z_{t+1}, is not pinned down in equilibrium. In particular, if there is an equilibrium in which some particular value of $\gamma = b_{t+1}/z_{t+1}$ holds, then there exists an equilibrium with the same allocations of consumption, output and capital, for all other values of γ.

(j) Show, using the linear homogeneity assumption on F, the firm's first order condition on labor, the entrepeneur's financing constraint, and the zero ex post profit condition, that

$$r_{t+1} = (F_{k,t+1} + 1 - \delta)(1 + \gamma) - \gamma R_{t+1}.$$

Use this to show, using your result in (v), that the equity premium, P_t, increases with γ.

(k) Let $u(c_t) = \log(c_t)$ and resource constraint, $c_t + k_{t+1} = k_t^{\alpha} n_t^{(1-\alpha)} \exp(x_t)$, where x_t has a first order autoregressive representation: $x_t = \rho x_{t-1} + \varepsilon_t$, where ε_t iid over time and independent of x_{t-1}.

Show that the equity premium, P_t, reduces to:

$$P_t = 1 - Cov_t(\exp(-\varepsilon_{t+1}), \exp(\varepsilon_{t+1}))(1 + \gamma),$$

where γ is the debt/equity ratio, assumed to be fixed.

Suppose ε_t is iid over time with $\varepsilon_t = \sigma$ with probability $1/2$ and $\varepsilon_t = -\sigma$ with probability $1/2$. It is easily verified that $Var(\varepsilon_t) = (\sigma)^2$, so that σ is the standard deviation of ε_t. Based on his analysis of quarterly U.S. data, Prescott (1986, Federal Reserve Bank of Minneapolis Quarterly Review) has argued that an empirically plausible value for this quantity is .00763. Compute P_t for the case $\gamma = 0$.

Mehra and Prescott argued that the equity premium in the U.S. averages 1.07 percent per annum, or, 1.017 per quarter. What value of γ is necessary to make P_t this large?

3. This question is designed to draw attention to the fact that the conditions underlying Blackwell's Theorem and the contraction mapping theorem are sufficient, and not necessary. Thus, even if some of these conditions are false, it may still be that aspects of the contraction mapping theorem go through.

   Consider the following functional equation, taken from the midterm:

$$T(v) = \max_{0 \le \kappa \le A+1-\delta} \frac{[A+1-\delta-\kappa]^{(1-\sigma)}}{1-\sigma} + \beta\kappa^{(1-\sigma)}v.$$

   Suppose $\sigma > 1$ and $\beta(A+1-\delta)^{1-\sigma} < 1$, as in the midterm.

   (a) Show: $T(v) = \infty$ for $v > 0$, $T(0) = \frac{[A+1-\delta]^{(1-\sigma)}}{1-\sigma}$.

   (b) Show: the derivative of T at $v = v_0 < 0$ is:

$$\frac{dT(v_0)}{dv} = \beta\kappa_0^{(1-\sigma)},$$

   where

$$\kappa_0 = \text{argmax}_{0 \le \kappa \le A+1-\delta} \frac{[A+1-\delta-\kappa]^{(1-\sigma)}}{1-\sigma} + \beta\kappa^{(1-\sigma)}v_0.$$

   (c) Show that the operator T is not a contraction mapping (hint: (i) think about its derivative for small negative v, and note the relationship between the contraction property and the slope of a function; and (ii) it is not enough to establish that T fails to satisfy one of Blackwell's sufficient conditions.)

   (d) Consider $v_0 = 0$ and the sequence $v_i = T(v_{i-1})$, for $i = 1, 2, 3, \ldots$. Does this converge to v, the value associated with the maximum in the sequence problem in the midterm?