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D11-2, Winter 1997

MIDTERM EXAM

There are four questions. The total number of possible points is 100, and the number of points per question is given in parentheses. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have 1 hour and 50 minutes. Good luck!

1. (40) Consider the following "Ak" economy. The resource constraint is:

$$c_t + k_{t+1} - (1-\delta)k_t \leq Ak_t.$$

Preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}, \quad \sigma > 0, \quad \sigma \neq 1.$$

The initial capital stock, $k_0 > 0$, is given. In addition, the following non-negativity constraints must be respected:

$$c_t, k_{t+1} \geq 0, \quad t = 0, 1, 2, \ldots.$$

Finally, suppose:

$$\beta(A + 1 - \delta)^{1-\sigma} < 1.$$

(a) (10) Show that this problem can be written:

$$V(k_0) = k_0 v,$$

where

$$v = \max_{\{0 \leq \lambda_t \leq A+1-\delta\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\prod_{i=0}^{t-1} \lambda_i \right)^{(1-\sigma)} \frac{[A + 1 - \delta - \lambda_t]^{(1-\sigma)}}{1-\sigma}.$$

Here, $\lambda_t = k_{t+1}/k_t$; $\prod_{i=0}^{t-1} \lambda_i \equiv 1$ for $t = 0$.

(b) (10) Prove that $-\infty < v < \infty$. (Hint: be careful to treat the cases, $\sigma < 1$ and $\sigma > 1$ separately.)

(c) (5) Explain why v defined above also solves:

$$v = T(v);$$

where

$$T(v) = \max_{0 \leq \lambda \leq A+1-\delta} \frac{[A + 1 - \delta - \lambda]^{(1-\sigma)}}{1-\sigma} + \beta \lambda^{(1-\sigma)} v.$$

(d) (10) Prove that T is a contraction mapping when $0 < \sigma < 1$ (hint: you may appeal to any theorems you learned in this course.)

(e) (5) Explain why T is not guaranteed to be a contraction mapping when $\sigma > 1$.

2. (30) A model is said to display "the convergence property" if the value of the initial capital stock has no long-run impact on the stock of capital. Consider the following economy:

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^\gamma}{\gamma}; \;\; \gamma < 1; \;\; c_t + k_{t+1} - (1-\delta)k_t \leq f(k_t; z_t); \;\; 0 < \delta < 1;$$

$$z_t = \exp(\mu) z_{t-1}; \;\; \mu > 0; \;\; t = 0; 1; 2; \ldots$$

In addition, $k_0 > 0$ is given.

(a) (10) Explain why it is that when

$$f(k_t; z_t) = k_t^\alpha z_t^{(1-\alpha)}; \;\; 0 < \alpha < 1;$$

then the model displays the convergence property.

(b) (10) Explain why it is that when

$$f(k_t; z_t) = A k_t; \;\; A > 0; \;\; \beta(A + 1 - \delta) > 0;$$

then the efficient allocations display growth, yet the model economy does not display the convergence property.

(c) (10) Explain why it is that when the above economy is modified so that

$$f(k_t; z_t) = A k_t + B k_t^\alpha; \;\; B > 0; \;\; 0 < \alpha < 1;$$

then the economy "sort of" displays the convergence property.

3. (20) Let $s^t = (s_0, s_1, ..., s_t)$ denote the history of exogenous shocks up to time t; where $s_t \in (s(1), ..., s(N))$; and N is a finite number. Let $\pi(s^t)$ denote the probability of history s^t. Let preferences be given by:

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) u\left(c(s^t)\right),$$

where $c(s^t)$ denotes consumption conditional on history s^t. Suppose that the usual assumptions apply to u; for example that it be strictly increasing. Feasibility of the allocations requires:

$$c(s^t) + i(s^t) \leq k(s^{t-1})^{\alpha} n(s^t)^{(1-\alpha)};\ c(s^t), k(s^t) \geq 0;\ 0 \leq n(s^t) \leq 1;\ \text{all } s^t;$$

$$i(s^t) = k(s^t) - (1-\delta)k(s^{t-1});$$

and the initial value of capital, $k(s^{-1})$, is given. Consider an Arrow-Debreu equilibrium for this economy, with the following market prices:

$p(s^t)$ price of history s^t output
$p(s^t)w(s^t)$ price of history s^t labor services
$p(s^t)r(s^t)$ price of history s^t capital services

(a) (10) Define formally an Arrow-Debreu equilibrium (hint: this requires first formally stating the household and firm problems).

(b) (5) Explain why all prices must be strictly positive in equilibrium.

(c) (5) Explain why the equilibrium allocations are efficient.

4. (10) Set up a recursive competitive equilibrium for the model economy in the previous question.