Christiano
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Rough Guide to ANSWERS TO FINAL EXAM

1. Answers to question 1

   (a) Let $\pi = (\delta, \tau)$ denote a government policy, and let $F(\pi) = (c_1(\pi), c_2(\pi), l(\pi), k(\pi))$, denote the equilibrium competitive allocations given policy π. A Ramsey equilibrium is a π^* and $F(\pi)$ where: (i) for any π, $F(\pi)$ maximizes the household's utility subject to its budget constraints, and (ii) π^* solves the problem, maximize, over π, $u(c_1(\pi)+c_2(\pi), l(\pi))$, subject to the government's budget constraint. The capital tax rate satisfies $(1-\delta)R = 1$. Any tax rate higher than this would result in $k = 0$, and so no revenues from the capital tax. Any tax rate lower than this would result in $k = \omega$, with revenues to the government equal to $\delta R\omega$. The marginal effect of raising δ from such a low level operates like a lump sum tax, and so the government would never settle for such a low tax rate. Also, $k = \omega$ and $c_1 = 0$. At the Ramsey tax rate, $(1-\delta)R = 1$, or $\delta = (R-1)/R$, so that government revenues from taxing capital total $\delta Rk = (R-1)\omega < G$ by assumption. Since the Ramsey tax on capital is not enough to fund all of government spending, the Ramsey labor tax rate must be positive.

   (b) At the end of period 1, after $k = \omega$, the government has an incentive to increase δ above its Ramsey value, so that $\delta R\omega = G$. Note that this implies a deviation *up* in the capital tax rate, since $(R-1)\omega < G \Rightarrow R-1 < G/\omega \Rightarrow (R-1)/R < G/(R\omega)$. Since by assumption, $R\omega > G$, $\delta \leq 1$ will work. Also, with this deviation from the Ramsey capital tax rate, it would be possible to set the labor tax rate to zero.

   To see that the government would raise utility by deviating, note that the household's first order condition for choosing labor is: $l = 1-\tau$. Under the Ramsey policy and under a deviation, $c_1 = 0$, $k = \omega$, $c_2 = (1-\delta)R\omega + (1-\tau)l = (1-\delta)R\omega + (1-\tau)^2$. Thus, the policy of deviating solves

$$\max_{\delta,\tau} u(c_1+c_2, l) = (1-\delta)R\omega + \frac{1}{2}(1-\tau)^2, \text{ s.t. } G = \delta R\omega + \tau(1-\tau).$$

The government budget constraint can be rewritten, $(1-\delta)R\omega = R\omega - G + \tau(1-\tau)$. Then, substituting out for δ into the government's objective:

$$\max_{\tau} R\omega - G + \tau(1-\tau) + \frac{1}{2}(1-\tau)^2,$$

or

$$\max_{\tau} R\omega - G + \frac{1}{2}(1-\tau^2).$$

This objective function is strictly decreasing in τ for $0 \leq \tau \leq 1$. Since $\tau > 0$ under the Ramsey plan, it follows that utility is increased by reducing the labor tax rate from its Ramsey value.

(c) A sustainable equilibrium is a collection of numbers and two functions, $\tau^*, \delta^*, \tilde{c}_1, c_2(\delta,\tau), l(\delta,\tau), \tilde{k}$, satisfying the following three properties: (i) the household problem is solved. That is, at date 1, $\tilde{c}_1, c_2(\delta^*,\tau^*), l(\delta^*,\tau^*), \tilde{k}$ solve the problem: max $u(c_1+c_2,l)$ over c_1, c_2, l, and k, subject to the period 1 and period 2 budget constraints and that the capital and labor tax rates are given by δ^*, τ^*. The functions, $c_2(\delta,\tau), l(\delta,\tau)$, solve for any δ, τ, the household's period 2 maximization problem: max over c_2, l, the problem $u(\tilde{c}_1+c_2,l)$ subject to $c_2 \leq (1-\delta)R\tilde{k}+(1-\tau)l$. (ii) δ^*, τ^* solve the government problem: maximize $u\left(\tilde{c}_1 + c_2(\delta,\tau), l(\delta,\tau)\right)$ over δ, τ, subject to the government budget constraint. In a sustainable equilibrium, $\delta = 1$. If $\tilde{k} > 0$, then $\delta = 1$ clearly is optimal since this maximizes revenues from what is a lump-sum tax. If $\tilde{k} = 0$, then all values of δ produce the same return for the government, as so $\delta = 1$ is optimizing in this case too.

2. Answer to question 2.

(a) Sequence of markets equilibrium. At each date, t, the household maximizes discounted utility from then on:

$$\sum_{j=t}^{\infty} \beta^{j-t} u(c_j),$$

subject to a sequence of budget constraints:

$$c_j + i_{pj} \leq r_j k_{pj} + w_j n - T_j, \; j \geq t,$$

where w_j and r_j are market prices beyond the control of the household. The household uses its entire endowment of time for labor effort, n, because it does not value leisure. The firms choose n_t and $k_{p,t}$ such that profits are maximized, where profits are defined as follows: 1-α

$$k_{g,t}^{\gamma} n_t^{(1-\alpha)} k_{pt}^{\alpha} - w_t n_t - r_t k_t.$$

A sequence of markets equilibrium is a set of prices and quantities, $\{r_t, w_t; t \geq 0\}$, $\{y_t, c_t, n, i_{pt}, i_{gt}; t \geq 0\}$ and taxes, $\{T_t; t \geq 0\}$ such that

i.
- given taxes and prices, the quantities solve the household problem.
- given the prices, the quantities solve the firm problem.
- given the quantities and a value of s, the government budget constraint is satisfied.
- the resource constraint is satisfied.

(b) the first order condition for the household is

$$u_{c,t} = \beta u_{c,t+1}[r_{t+1} + 1 - \delta_p],$$

and the firm sets $f_{k_p,t+1} = r_{t+1}$, where $f_{k_p,t+1}$ is the marginal product of private capital. Combining these, and taking functional forms into account:

$$\left(\frac{c_{t+1}}{c_t}\right)^{\nu} = \beta[\alpha\left(\frac{nk_{g,t+1}}{k_{p,t+1}}\right)^{(1-\alpha)} + 1 - \delta_p].$$

From this one can see that when $\gamma = 1 - \alpha$, then public capital acts to lift up the return on private capital, so that the incentive to invest doesn't die along a growth path. When $\gamma = 0$,then diminishing returns acts to eventuall kill the incentive to investment along the growth path. Let g_c denote the gross growth rate of consumption in a balanced growth path. Then,

$$(g_c)^{\nu} = \beta[\alpha(ns)^{(1-\alpha)} + 1 - \delta_p].$$

Suppose g_c corresponds to some given positive net growth rate, ie., $g_c > 1$. Then,

$$s = \frac{1}{n}\left\{\frac{1}{\alpha}\left[\frac{g_c^{\nu}}{\beta} + \delta_p - 1\right]\right\}^{\frac{1}{1-\alpha}}.$$

The number in square brackets is positive, so that s is well defined. Thus the Euler equation is consistent with constant consumption growth in steady state. To fully answer the question, we need to establish (i) that the other equations - the household budget equation and the resource constraint - are also satisfied with a constant consumption growth rate and (ii) that the other quantity variables display positive growth too. Let g_g and g_p denote the gross growth rates of government and private capital, respectively. Then, the government's policy for choosing $k_{g,t}$ implies:

$$g_g = g_p = g,$$

say. Note that output can be written

$$k_{gt}^{(1-\alpha)} k_{pt}^{\alpha} n^{(1-\alpha)} = k_{gt}(k_{pt}/k_{gt})^{\alpha} n^{(1-\alpha)} = k_{gt} s^{\alpha} n^{(1-\alpha)}.$$

Divide the resource constraint by k_{gt}:

$$\frac{c_t}{k_{gt}} + g_{t+1} - (1-\delta_g) + g_{t+1} - (1-\delta_p) = s^{\alpha} n^{(1-\alpha)}.$$

So, in a constant growth steady state (i.e., $g_{t+1} = g$, constant) the consumption to public capital ratio is a constant, equal to the following:

$$s^{\alpha} n^{(1-\alpha)} + (1-\delta_g) + (1-\delta_p) + 2g.$$

But, the consumption to public capital ratio being constant implies:

$$g_c = g.$$

The household budget constraint is trivially satisfied, since it is equivalent with the resource constraint given the first order conditions of firms, linear homogeneity of the production function with respect to firms' choice variables, and the government budget constraint.

(c) The planner's problem is: choose $c_t, k_{g,t+1}, k_{p,t+1}$, $t \geq 0$ to maximize discounted utility. After substituting out consumption using the resource constraint, the problem becomes:

$$\max_{\{k_{g,t+1}, k_{p,t+1}\}} \sum_{t=0}^{\infty} \beta^t u[k_{gt}^{(1-\alpha)} n^{(1-\alpha)} k_{pt}^{\alpha} + (1-\delta_g)k_{g,t} + (1-\delta_p)k_{p,t}$$
$$-k_{p,t+1} - k_{g,t+1}],$$

subject to the object in square brackets (consumption) being non-negative at all dates, and to $k_{g,t}, k_{p,t} \geq 0$. The planner's first order conditions are:

$$\begin{aligned} u_{c,t} &= \beta u_{c,t+1}[f_{k_p,t+1} + 1 - \delta_p] \\ u_{c,t} &= \beta u_{c,t+1}[f_{k_g,t+1} + 1 - \delta_g], \end{aligned}$$

for $t = 0, 1, 2, ...$ With the functional forms:

$$\begin{aligned} \left(\frac{c_{t+1}}{c_t}\right)^{\nu} &= \beta[\alpha k_{g,t+1}^{\gamma} \left(\frac{n}{k_{p,t+1}}\right)^{(1-\alpha)} + 1 - \delta_p] \\ \left(\frac{c_{t+1}}{c_t}\right)^{\nu} &= \beta[\gamma (k_{g,t+1})^{\gamma-1} n^{(1-\alpha)} \left(k_{p,t+1}\right)^{\alpha} + 1 - \delta_g]. \end{aligned}$$

Substituting out consumption using the resource constraint, these two equations represent a vector difference equation in k, k', k'', where $k = [k_g\ k_p]'$. There are many solutions to this equation that are consistent with the given initial condition, $k_0 = [k_{g,0}\ k_{p,0}]$. One can construct the whole family of solutions by indexing them by k_1: different values of k_1 give rise, by iterating on the euler equation, to different sequences of capital. Not all are optimal. Only the one solution that also satisfies the transversality condition is optimal. Thus, satisfying the Euler equation is not sufficient for an optimum.

(d) Setting $\gamma = 1 - \alpha$ and equating the planner's two first order conditions, we get:

$$\begin{aligned} & \beta[\alpha \left(\frac{n k_{g,t+1}}{k_{p,t+1}}\right)^{(1-\alpha)} + 1 - \delta_p] \\ = \ & \beta[(1-\alpha) n^{(1-\alpha)} \left(\frac{k_{p,t+1}}{k_{g,t+1}}\right)^{\alpha} + 1 - \delta_g], \end{aligned}$$

which requires that $\frac{k_{p,t+1}}{k_{g,t+1}}$ be a particular constant for $t = 0, 1, ...$ Call this constant s^*. By setting $s = s^*$ the government cannot do better, since this achieves the planner's optimum.

3. Question 3. First part...go to non-unit elasticity of substitution between capital and labor in the production function for consumption goods.