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D11-2, Winter 1998

MIDTERM EXAM

There are four questions. The total number of possible points is 100, and the number of points per question is given in parentheses. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have 1 hour and 50 minutes. Good luck!

1. (28) Suppose a household's preferences are given by $\sum_{t=0}^{\infty} \beta^t u(c_t)$, where u is strictly concave, increasing and differentiable. Suppose the household faces the following sequence of budget constraints:

$$c_t + p_t x_t \leq r_t k_t + w_t + \pi_t, \ t = 0, 1, 2, ...,$$

where

$$x_t = k_{t+1} - (1-\delta) k_t.$$

Here, $k_0 > 0$ is given. The household also takes the non-negative sequence of prices and profits, p_t, w_t, π_t, $t \geq 0$, as given. (Here, w_t is the wage rate, and the household is assumed to supply one unit of labor inelastically to the market.) The household's non-negativity constraints are $k_{t+1}, c_t \geq 0$ for $t \geq 0$. Finally, suppose that prices and profits satisfy the following *boundedness property*: if a sequence, $\{c_t\}_{t=0}^{\infty}$, is consistent with the budget and non-negativity constraints, then $c_t < \bar{c}$ for all $t \geq 0$, where $\bar{c} < \infty$.

Suppose $\left\{k_{t+1}^*, c_t^*\right\}_{t=0}^{\infty}$ satisfy the budget and non-negativity constraints. Suppose they also satisfy:

$$p_t u_c(c_t^*) = \beta u_c(c_{t+1}^*) \left[r_{t+1} + (1-\delta) p_{t+1}\right],$$

where u_c is the derivative of u with respect to c, and

$$lim_{T \to \infty} \beta^T u_c(c_T^*) \left[r_T + (1-\delta) p_T\right] k_T = 0.$$

Show that there is no other $\{k_{t+1}, c_t\}_{t=0}^{\infty}$ that satisfies the budget and non-negativity constraints, which generates higher utility than $\left\{k_{t+1}^*, c_t^*\right\}_{t=0}^{\infty}$. What role does the boundedness property play in your argument?

2. (28) Consider the following two-sector model of optimal growth. A planner seeks to maximize the utility of the representative agent given by $\sum_{t=0}^{\infty} \beta^t u(c_t)$, where c_t is consumption of good 1 at t. Sector 1 produces consumption goods using capital, k_{1t}, and labor, n_{1t}, according to the production function, $c_t \leq f_1(k_{1t}, n_{1t})$. Sector 2 produces the capital good according to the production function $k_{t+1} \leq f_2(k_{2t}, n_{2t})$. The constraint on labor is $n_{1t} + n_{2t} = 1$, where 1 denotes the total amount of labor supplied. The other constraints include $n_{it}, k_{it} \geq 0$, $i = 1, 2$, and $k_{t+1} \geq 0$. The sum of the amounts of capital used in each sector cannot exceed the initial capital in the economy, that is, $k_{1t} + k_{2t} \leq k_t$, and $k_0 > 0$, given.

   Let $v(k_0)$ be the function which characterizes how the maximized value of the representative agent's discounted utility varies with k_0. Show that v solves a particular functional equation. (Hint: if you use the notation from Stokey and Lucas, you must define a function, $F(k, k')$ and a feasibility set, $\Gamma(k)$.)

3. (16) A recent AER paper analyzes the following model of investment-specific technical change:

$$k_{t+1} - (1 - \delta)k_t = q_t x_t, \;\; c_t + q_t x_t \leq f(k_t),$$

   where f strictly concave and $f'(k) \to 0$ as $k \to \infty$, $f'(k) \to \infty$ as $k \to 0$. (There is no typo here, $q_t x_t$ appears in both equations.) Suppose $q_t = exp(\mu t)$, $\mu > 0$. Suppose the allocations are chosen to maximize the present discounted value of the utility of consumption, where the utility function satisfies all the usual assumptions.

   (a) What is the date t consumption cost of capital, $P_{k',t}$? Does it exhibit a trend?

   (b) Display an expression for the rate of return on capital. Does it involve q_t?

   (c) Do the optimal allocations in this economy converge to a steady state growth path with positive growth? How is this growth rate influenced by μ?

4. (28) Consider a household which solves the following problem:

$$v(k,r,w) = \max_{c,l \in B(k,r,w)} u(c,l),$$

where $u : \Re^2_+ \to \Re$ is a strictly concave, twice continuously differentiable, strictly increasing function in its two arguments: consumption, c, and leisure, l. The constraints the household must obey in selecting c, l are summarized by B:

$$B(k,r,w) = \{c, l : 0 \leq c \leq rk + w(1-l),\ 0 \leq l \leq 1\}.$$

Here, $r > 0$ is the market rental rate on capital and $w > 0$ is the market wage rate, neither of which the household can control. Also, $k > 0$ is the household's stock of capital. Prove that the derivative of v with respect to k exists, and display a formula for it. If you make use of a theorem to help prove your result, be sure to state it clearly.