1. (28) Suppose a household’s preferences are given by $\sum_{t=0}^{\infty} \beta^t u(c_t)$, where $u$ is strictly concave, increasing and differentiable. Suppose the household faces the following sequence of budget constraints:

$$c_t + p_t x_t \leq r_t k_t + w_t + \pi_t, \quad t = 0, 1, 2, \ldots,$$

where $x_t = k_{t+1} - (1 - \delta) k_t$. 

Here, $k_0 > 0$ is given. The household also takes the non-negative sequence of prices and profits, $p_t, w_t, \pi_t, t \geq 0$, as given. (Here, $w_t$ is the wage rate, and the household is assumed to supply one unit of labor inelastically to the market.) The household’s non-negativity constraints are $k_t + 1, c_t \geq 0$ for $t \geq 0$. Finally, suppose that prices and profits satisfy the following boundedness property: if a sequence, $\{c_t\}_{t=0}^{\infty}$, is consistent with the budget and non-negativity constraints, then $c_t < \bar{c}$ for all $t \geq 0$, where $\bar{c} < \infty$.

Suppose $\{k_t + 1, c_t\}_{t=0}^{\infty}$ satisfy the budget and non-negativity constraints. Suppose they also satisfy:

$$p_t u_c(c_t^*) = \beta u_c(c_{t+1}^*) \left[ r_{t+1} + (1 - \delta) p_{t+1} \right],$$

where $u_c$ is the derivative of $u$ with respect to $c$, and

$$\lim_{T \to \infty} \beta^T u_c(c_T^*) \left[ r_T + (1 - \delta) p_T \right] k_T = 0.$$

Show that there is no other $\{k_t + 1, c_t\}_{t=0}^{\infty}$ that satisfies the budget and non-negativity constraints, which generates higher utility than $\{k_t + 1, c_t\}_{t=0}^{\infty}$. What role does the boundedness property play in your argument?
2. (28) Consider the following two-sector model of optimal growth. A planner seeks to maximize the utility of the representative agent given by
\[ \sum_{t=0}^{\infty} \beta^t u(c_t), \]
where \( c_t \) is consumption of good 1 at \( t \). Sector 1 produces consumption goods using capital, \( k_{1t} \), and labor, \( n_{1t} \), according to the production function, \( c_t \leq f_1(k_{1t}, n_{1t}) \). Sector 2 produces the capital good according to the production function \( k_{t+1} \leq f_2(k_{2t}, n_{2t}) \). The constraint on labor is \( n_{1t} + n_{2t} = 1 \), where 1 denotes the total amount of labor supplied. The other constraints include \( n_{it}, k_{it} \geq 0, i = 1, 2 \), and \( k_{t+1} \geq 0 \). The sum of the amounts of capital used in each sector cannot exceed the initial capital in the economy, that is, \( k_{1t} + k_{2t} \leq k_t \), and \( k_0 > 0 \), given.

Let \( v(k_0) \) be the function which characterizes how the maximized value of the representative agent’s discounted utility varies with \( k_0 \). Show that \( v \) solves a particular functional equation. (Hint: if you use the notation from Stokey and Lucas, you must define a function, \( F(k, k') \) and a feasibility set, \( \Gamma(k) \).)

3. (16) A recent AER paper analyzes the following model of investment-specific technical change:
\[ k_{t+1} = (1 - \delta)k_t - q_t x_t, \quad c_t + q_t x_t \leq f(k_t), \]
where \( f \) strictly concave and \( f'(k) \to 0 \) as \( k \to \infty \), \( f'(k) \to \infty \) as \( k \to 0 \). (There is no typo here, \( q_t x_t \) appears in both equations.) Suppose \( q_t = \exp(\mu t), \mu > 0 \). Suppose the allocations are chosen to maximize the present discounted value of the utility of consumption, where the utility function satisfies all the usual assumptions.

(a) What is the date \( t \) consumption cost of capital, \( P_{0,t} \)? Does it exhibit a trend?
(b) Display an expression for the rate of return on capital. Does it involve \( q_t \)?
(c) Do the optimal allocations in this economy converge to a steady state growth path with positive growth? How is this growth rate influenced by \( \mu \)?

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4. (28) Consider a household which solves the following problem:

\[ v(k, r, w) = \max_{c, l \in B(k, r, w)} u(c, l), \]

where \( u : \mathbb{R}_+^2 \to \mathbb{R} \) is a strictly concave, twice continuously differentiable, strictly increasing function in its two arguments: consumption, \( c \), and leisure, \( l \). The constraints the household must obey in selecting \( c, l \) are summarized by \( B \):

\[ B(k, r, w) = \{ c, l : 0 \leq c \leq rk + w(1 - l), \ 0 \leq l \leq 1 \}. \]

Here, \( r > 0 \) is the market rental rate on capital and \( w > 0 \) is the market wage rate, neither of which the household can control. Also, \( k > 0 \) is the household’s stock of capital. Prove that the derivative of \( v \) with respect to \( k \) exists, and display a formula for it. If you make use of a theorem to help prove your result, be sure to state it clearly.