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D11-2, Winter 1998

Rough Guide to MIDTERM EXAM ANSWERS

1. Let
$$U_t(k_t, k_{t+1}) = u\left(r_t k_t + w_t + \pi_t - p_t\left[k_{t+1} - (1-\delta)k_t\right]\right).$$
By strict concavity:
$$U_t(k_t, k_{t+1}) \leq U_t^* + U_{t,1}^*(k_t - k_t^*) + U_{t,2}^*(k_{t+1} - k_{t+1}^*),$$
where k_t, k_{t+1} is any other sequence that is consistent with the non-negativity and budget constraints. Also, $U_t^* = U_t(k_t^*, k_{t+1}^*)$ and $U_{t,i}^*$ is the derivative of U_t^* with respect its i^{th} argument, $i = 1, 2$. We want to show:
$$D = lim_{T\to\infty}\sum_{t=0}^{T}\beta^t\left[U_t^* - U_t(k_t, k_{t+1})\right] \geq 0.$$
The boundedness condition assures that the above limit converges, so evaluating D is a meaningful enterprise. The result follows by literally copying the proof in S-L, Theorem 4.15.

2. Let,
$$F(k, k') = max_{(n_1,n_2,c,k_1,k_2)\in B(k,k')}u(c),$$
where
$$\begin{aligned} B(k, k') &= \{n_1, n_2, k_1, k_2, c : c \leq f_1(k_1, n_1),\ k' \leq f_2(k_2, n_2),\ n_1 + n_2 = 1 \\ &\quad k_1 + k_2 \leq k,\ k_i \geq 0,\ n_i \geq 0,\ i = 1, 2\}. \end{aligned}$$
Then, the sequence problem can be written
$$\begin{aligned} v(k_0) &= max_{\{k_{t+1}\}_{t=0}^{\infty}}\sum_{t=0}^{\infty}\beta^t F(k_t, k_{t+1}) \\ &= max_{k_1}\left\{F(k_0, k_1) + \beta max_{k_2,k_3,...}\left[F(k_1, k_2) + \beta F(k_2, k_3) + ...\right]\right\} \\ &= max_{k_1}\left\{F(k_0, k_1) + \beta v(k_1)\right\}. \end{aligned}$$
So, v solves the following functional equation:
$$v(k) = max_{k'\in\Gamma(k)}F(k, k') + \beta v(k').$$

3. (See equations (3), (5) and the second equality in (7), in Hulten, AER, September 1992.) This setup is seen to be identical to the standard neoclassical growth model, when $q_t x_t$ is replaced by i_t in both places. So, the behavior of q_t is, in fact, completely irrelevant. It is $i_t = q_t x_t$ that is uniquely determined. Whatever q_t is, x_t will adjust so that $x_t = i_t/q_t$.

   The date t consumption price of a unit of k_{t+1} is found as follows. One unit of k_{t+1} requires one unit of $q_t x_t$. One unit of $q_t x_t$ requires giving up one unit of c_t. So, $P_{k',t} = 1$. Of course, this is trendless.

   The extra consumption you get in the next period from investing in an extra unit of capital today is (i) the extra amount of goods you get, $f'(k_{t+1})$, and (ii) the amount by which you can reduce investment in $t+1$ to keep k_{t+2} unchanged. This reduction in investment, $q_{t+1}x_{t+1}$, is $(1-\delta)$. The reduction in investment of $1-\delta$ allows you to expand consumption in $t+1$ by $1-\delta$. So the date $t+1$ payoff from an extra unit of k_{t+1} is $f'(k_{t+1}) + 1 - \delta$. This is the rate of return on capital too, since $P_{k',t} = 1$. There is no reason for q_t to enter here.

   Since this is the standard growth model in disguise, consumption and capital eventually converge monotonically to constant values. There is no growth in steady state, and μ is completely irrelevant.

4. My answer to this question is much longer than you could feasibly do in an exam situation. I will use two results to answer the question: the Theorem of the Maximum, and the Benveniste and Scheinkman theorem. The proof style will mimick that of Theorem 4.11 in the book. That theorem does not apply literally, since it is applied in a different environment. Still, the basic strategy works.

   The Benveniste and Scheinkman theorem states: Suppose $X \subseteq \Re^k$ convex, $V : X \to \Re$ concave, $x_0 \in int(X)$, and D is a neighborhood of x_0. Also, W is (i) concave and differentiable on D, (ii) $W(x_0) = V(x_0)$, (iii) $W(x) \leq V(x)$, $x \in D$. Then, V is differentiable at x_0 with $V'(x_0) = W'(x_0)$.

   It is convenient to exploit the strictly increasing property of the utility function so that $c = w(1-l) + rk$. Substitute this into the utility function, and the problem becomes

$$v(k) = \max_{0 \le l \le 1} u\left(w(1-l) + rk, l\right),$$

where I dropped r and w from v for convenience. The existence of v is guaranteed by the continuity of u and by the fact that the constraint set is non-empty and compact. To establish concavity of v, consider $k_1, k_2, k_\theta > 0$, where $k_\theta = \theta k_1 + (1-\theta)k_2$ for $\theta \in (0,1)$. Also, let l_1, l_2 be the leisure decision when $k = k_1$, k_2, and let $l_\theta = \theta l_1 + (1-\theta)l_2$. (Note, l_θ is *not* necessarily the leisure decision taken when $k = k_\theta$.) Then,

$$\begin{aligned}
&\theta v(k_1) + (1-\theta)v(k_2) \\
&= \theta u(w(1-l_1) + rk_1, l_1) + (1-\theta)u(w(1-l_2) + rk_2, l_2) \text{ (by def. of } l_i, k_i, i = 1, 2) \\
&\le u(w(1-l_\theta) + rk_\theta, l_\theta) \text{ (by concavity of } u) \\
&\le v(k_\theta) \text{ (by the fact that } v(k_\theta) \text{ is the maximized)},
\end{aligned}$$

so that concavity is established.

The remainder of the proof mimicks the strategy in the proof of Theorem 4.11, which applies the Benveniste-Scheinkman theorem to a problem different from the present one. Let l^0 be the chosen value of l when $k = k^0$, for some k^0. Then, there is a neighborhood, D, about k^0 such that l^0 is feasible for all $k \in D$. Define $W : D \rightarrow \Re$:

$$W(k) = u\left(w(1-l^0) + rk, l^0\right) \le v(k).$$

The weak inequality holds for all $k \in D$, with equality for $k = k^0$. The conditions of the B-S theorem are satisfied. As a result, the derivative of v exists at $k = k^0$, and that derivative is $u_c\left(w(1-l^0) + rk^0, l^0\right) r$.