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D11-1, Fall 1998

FINAL EXAM

There are 5 questions. The total number of possible points is 100, and the number of points per question is given in parentheses. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have 2 hours. Good luck!

1. (30) Consider the problem of the representative household:

$$\max_{\{c_t, B_{t+1}; t \geq 0\}} \sum_{t=0}^{\infty} \beta^t u(c_t),\ 0 < \beta < 1,$$

where $u : R_+ \to R$ is strictly concave, increasing, and differentiable. The budget constraint, for $t \geq 0$, is:

$$c_t + \frac{B_{t+1}}{1+R} \leq y - \tau + B_t,\ B_0 > 0,\ \text{given.}$$

Here, B_t, c_t, y, τ, R denote, respectively, bond holdings, consumption, income, taxes, and the rate of interest. As the notation indicates, y, τ, and R are assumed to be constant. They are beyond the control of the household. Also, suppose that $1 + R = 1/\beta$. The nonnegativity constraints are c_t, $B_{t+1} \geq 0$ for all t.

There is a government which issues bonds, b_t. Its budget constraint, in per capita terms, is:

$$\frac{b_{t+1}}{1+R} + \tau = b_t,$$

where b_0 is given and equal to B_0. The left side of the above equality represents the government's source of funds - from issuance of new debt and taxes - and the right side represents the sole use of funds: paying off old debt.

Clearing in the goods and bonds market requires:

$$c_t = y,\ b_{t+1} = B_{t+1},\ t = 0, 1, 2, \ldots$$

(a) Show that if $b_0 > b^*$, where $b^* = (1+R)\tau/R$, then $b_t/[(1+R)^t] \to \Delta$ as $t \to \infty$, where $\Delta > 0$. Display an explicit formula for Δ.

(b) Write down a sequence of Euler equations and a transversality condition, and show that if a sequence, $c_t, B_{t+1},\ t > 0$, solves these, then they solve the household problem.

(c) Suppose $b_0 = B_0 > b^*$. Consider the following candidate solution to the household problem: $c_t = y$, $B_{t+1} = b_{t+1}$ for all t.

   i. Show that this sequence satisfies the budget and Euler equations, but *not* the transversality condition.

   ii. Prove that the candidate solution is in fact not optimal. (For example, you could identify another sequence that is feasible - i.e., consistent with the household's budget constraint and nonnegativity constraints - that generates higher utility.)

(d) Define a sequence of markets equilibrium for this economy. Explain why it is that, if there is an equilibrium, $b_0 = b^*$.

2. (10) Consider an economy in which, at date 0, the preferences of the representative household are:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t),$$

where u satisfies the usual assumptions, c_t denotes consumption, and n_t denotes hours worked. The resource constraint is:

$$c_t + I_{t+1} - I_t + k_{t+1} - (1-\delta)k_t \le f(k_t, n_t),$$

where f is the production function, which satisfies the usual assumptions, and δ denotes the rate of depreciation on capital. Also, k_t denotes the beginning-of-period t stock of capital, and I_t denotes the beginning-of-period t stock of inventories. Inventories represent a simple storage technology: put a unit of the output good into inventories at time t and it is available at the beginning of time $t+1$ for consumption.

(a) Describe a sequence of markets competitive decentralization for this economy. In your decentralization, suppose that households own the stocks of capital and inventories.

(b) Let the rate of return on capital from period t to period $t+1$ be denoted r_t^k. Suppose you observe a time series of data on this economy, in which $r_t^k > 1$ for $t = 1, ..., T$. What must the value of I_{t+1} be, for $t = 1, ..., T$? Explain carefully.

3. (25) Consider the following one period economy, which is composed of a continuum of identical households and a government. The representative household's utility is:

$$u(c_1 + c_2, l) = c_1 + c_2 - \frac{1}{2}l^2,$$

where c_i denotes consumption in the morning and the afternoon for $i = 1, 2$, respectively. Also, l denotes employment, which occurs in the morning only. The morning budget constraint is

$$c_1 + k \leq \omega$$

and the afternoon budget constraint is:

$$c_2 \leq R(1 - \delta)k + (1 - \tau)l,$$

where the wage rate is normalized to unity, R represents the gross rental rate on capital, δ represents the tax rate on capital, and τ represents the tax rate on labor. With the given specification of preferences, it is possible for the household to be in a situation where it is indifferent over all k such that $0 \leq k \leq \omega$. Suppose that, if they find themselves in such a situation, they always pick $k = \omega$.

The government has a fixed revenue requirement, G, and faces the following budget constraint in the afternoon:

$$G \leq \delta R k + \tau l,$$

where

$$(R - 1)\omega < G < \frac{1}{4}.$$

The government announces values for δ and τ at noon, after the decision about k and c_1 has already been made. The government is benevolent, in that it seeks to maximize the utility of private households.

(a) Define a sustainable equilibrium for this economy.

(b) Show that $k = 0$ in a sustainable equilibrium.

(c) Consider a version of this economy that repeats itself indefinitely. Suppose that at date t, $t = 0, 1, 2, \ldots$ the utility of the household is given by

$$\sum_{j=0}^{\infty} \beta^j u(c_{1,t+j} + c_{2,t+j}, l_{t+j}).$$

The order of events in date t is precisely what it is in the one period economy: first, the household makes its date t saving decision; then the government announces the date t tax rates, and then the date t labor decision is determined. Capital depreciates entirely between the afternoon of one day and the morning of the next. Define a sustainable equilibrium for this economy.

(d) Explain why it is that, for β sufficiently large, it is likely that there is an equilibrium in which the outcome is $k = \omega$ in each period. Explain why the only sustainable equilibrium in the finite horizon version of this economy involves $k = 0$ in each period.

4. (25) Consider an economy in which the representative household has the following preferences:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, n_t), \ 0 < \beta < 1,$$

where u satisfies the usual restrictions. The resource constraint is:

$$c_t + k_{t+1} - (1 - \delta)k_t \leq y_t,$$

and $0 < \delta < 1$. Final goods, y_t, are produced using the linear homogeneous technology:

$$y_t = \left[\int_0^1 x_t(i)^\lambda di \right]^{\frac{1}{\lambda}}, \ \lambda > 1,$$

where $x_t(i)$ is the quantity of the i^{th} intermediate good used. The technology for producing intermediate goods is

$$x_t(i) = k_t(i)^\mu n_t(i)^\gamma, \ \mu, \gamma > 0, \ 1 < \mu + \gamma \leq \psi \text{ for all } i \in (0, \infty).$$

Here, ψ is a parameter to be discussed below.

(a) Decentralize this economy and define a symmetric, sequence of markets equilibrium. In the equilibrium, give the final good technology to a representative competitive firm; give the technology for producing the i^{th} intermediate good to a monopolist; and suppose the representative household rents capital and labor in homogeneous, competitive markets. Only consider symmetric equilibria, in which all intermediate good firms behave identically. Let $p_t(i)$ denote the price of the i^{th} intermediate good and let w_t and r_t denote the wage rate and capital rental rate, respectively. All date t prices are denoted in units of the date t consumption good.

(b) Derive an expression for the demand curve faced by the monopolist. What happens to the slope of the demand curve (with $p_t(i)$ on the vertical axis and $x_t(i)$ on the horizontal) as $\lambda \to 1$? Provide intuition.

(c) What restriction on ψ is necessary if the monopolist is to have a well-defined maximization problem?

(d) Derive an expression for the markup (*i.e.*, the ratio of price to marginal cost) charged by the typical monopolist.

(e) Derive a simple expression for the level of profits in a symmetric equilibrium.

(f) Show that a necessary condition for allocations to constitute an interior equilibrium, is that they satisfy:

$$u_{c,t} = \beta u_{c,t+1} \left[\zeta \frac{y_{t+1}}{k_{t+1}} + 1 - \delta \right], \quad \frac{-u_{n,t}}{u_{c,t}} = \xi \frac{y_t}{n_t}, \ \ t = 0, 1, 2, ... \ ,$$

where $u_{c,t}$ and $u_{n,t}$ are the derivatives of u with respect to the first and second arguments. Derive expressions relating ζ and ξ to the model parameters.

5. (10) Consider an economy with capital of different vintages. At time t, the amount of capital of vintage τ, $k_{t,\tau}$, $\tau = 1, 2, 3, ...$, is

$$k_{t,\tau} = \gamma^{t-\tau}(1-\delta)^{\tau-1} i_{t-\tau},$$

where $\gamma > 1$, $0 < \delta < 1$, $i_{t-\tau}$ is the amount of investment, in time $t-\tau$ consumption units, applied in period $t-\tau$. Capital which has vintage

τ in period t has vintage $\tau+1$ in period $t+1$. Investment expenditures at time t, i_t, must all be applied to the latest vintage and results in $k_{t+1,1} = \gamma^t i_t$ units of new-vintage period $t+1$ installed capital goods. Consider a given amount of investment, i. Note that this investment applied in period $t+1$ produces more new-vintage installed capital (i.e., $\gamma^{t+1} i$) than the same level of investment applied in period t (i.e., $\gamma^t i$). This reflects the assumption, $\gamma > 1$ which is designed to capture the notion that there is exogenous technical progress that is embodied in new capital equipment. Note that the efficiency of a particular vintage stays constant over time, it's just that the efficiency of each succeeding vintage is greater than the efficiency of the previous one.

Capital of each vintage is operated with labor to produce a homogeneous output good, $y_{t,\tau}$ according to the following production function:

$$y_{t,\tau} = k_{t,\tau}^{\alpha} n_{t,\tau}^{1-\alpha}, \ 0 < \alpha < 1, \ \tau = 1, 2, 3, \dots .$$

Suppose there is a competitive market in capital of different vintages and in labor. Each vintage of capital has the same rental rate, r_t, since capital is measured in common efficiency units. Similarly, the wage rate is w_t.

(a) Show that a firm's profit maximizing choice of $n_{t,\tau}$ gives rise to the following relationships:

$$y_t = k_t^{\alpha} n_t^{1-\alpha}, \ (1-\alpha)\left(\frac{k_t}{n_t}\right)^{\alpha} = w_t, \ \alpha\left(\frac{k_t}{n_t}\right)^{\alpha-1} = r_t,$$

where

$$y_t = \sum_{\tau=1}^{\infty} y_{t,\tau}, \ k_t = \sum_{\tau=1}^{\infty} k_{t,\tau}, \ n_t = \sum_{\tau=1}^{\infty} n_{t,\tau}.$$

(b) Show that 'aggregate capital', k_t, evolves according to:

$$k_{t+1} = (1-\delta)k_t + \gamma^t i_t.$$