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Rough Guide to FINAL EXAM ANSWERS

1. (a) The budget equation of the government can be rewritten,

$$b_{t+1} = b^* + (1+R)(b_t - b^*),$$

so that

$$b_t = b^* + (1+R)^t (b_0 - b^*).$$

Then,

$$\frac{b_t}{(1+R)^t} \to b_0 - b^* = \Delta.$$

When $b_0 > b^*$, then obviously $\Delta > 0$.

(b) Write

$$F(B_t, B_{t+1}) = u(y - \tau + B_t - \frac{1}{1+R} B_{t+1}).$$

Note that F is strictly concave. The Euler equation is

$$F_2(B_t, B_{t+1}) + \beta F_1(B_{t+1}, B_{t+2}) = 0,$$

and the transversality condition is:

$$\lim_{T \to \infty} \beta^T u'(c_T) B_T \to 0.$$

The remainder of the proof proceeds exactly as in Stokey-Lucas, Thm 4.15.

i. From the government's budget constraint, if the household sets $B_t = b_t$ for all t, then

$$c_t = y - \tau + b_t - \frac{b_{t+1}}{1+R} = y - \tau + \tau = y,$$

so that the household's budget equation is satisfied. It is trivial to verify that the Euler equation is satisfied as a consequence of the facts, c_t is a constant, and $\beta = 1/(1+R)$. The transversality is not satisfied because $\beta^T B_T \to b_0 - b^* > 0$.

Question 1 Consider the strategy whereby the household sets $B_{t+1} = b^*$ for $t = 0, 1, 2, ...$. Note that with this strategy,

$$B_t - \frac{B_{t+1}}{1+R} = \frac{1+R}{R}\tau\left[1 - \frac{1}{1+R}\right] = \tau,\ t = 1, 2, 3, ...$$

Also,

$$B_0 - \frac{B_1}{1+R} = \Delta + b^* - \frac{b^*}{1+R} = \Delta + \tau.$$

Under this debt strategy, the budget constraint implies the following consumption sequence:

$$\begin{aligned} c_0 &= y - \tau + \Delta + \tau = y + \Delta \\ c_t &= y,\ t = 1, 2, 3, ... \end{aligned}$$

This is clearly better than the candidate sequence, since consumption is increased in period 0 without reducing it in any other date.

Here is some intuition for understanding what is going on here. The initial debt, B_0, can be split into two parts, $B_0 = b^* + (B_0 - b^*)$, or

$$B_0 = b^* + \Delta.$$

The first part, b^*, will eventually be paid off[1], whereas the second part is simply rolled over each period. The interest rate is such that the household is willing to purchase the first part. It prefers to sell out the first part and consume the proceeds immediately, to just rolling it over forever.

(c) A sequence of markets equilibrium is a set of quantities, $\{b_{t+1}, B_{t+1}, c_t; t \geq 0\}$ such that $\{b_{t+1}, c_t; t \geq 0\}$ solves the household problem, $\{B_{t+1}; t \geq 0\}$ is consistent with the government's budget equation, and goods

[1]This is true in the following sense. When $B_0 = b^*$, then repeated substitution with the government's budget constraint yields the result:

$$B_0 = \sum_{t=0}^{\infty} \frac{\tau}{(1+R)^t},$$

which says that the present value of tax receipts equals the current outstanding debt. Note that this does not require literally $B_t \to 0$ as $t \to \infty$.

and bond market clearing occurs. When $B_0 = b^*$, then the unique sequence of B_t's that solves the government's budget constraint does not solve the household problem.

Question 3 (c) Let the first date be $t = 0$, the second, $t = 1$, and so on. In the multiperiod setup, expectations can potentially depend on the past history of government actions. We therefore need a notation for this. Let the government actions at date t be denoted by $\pi_t = (\delta_t, \tau_t)$, for $t = 0, 1, 2, ...$, and let the history of government actions be denoted

$$\pi^t = (\pi_0, ..., \pi_t).$$

Expectations at date 0 are a couple of numbers, $\pi^* = (\delta^*, \tau^*)$, because there is no history of past actions. Write this as $X_0 = \pi^*$. At date 1 the expectation function is $X_1(\pi^0)$, at date 2 it's $X_2(\pi^1)$, and so on. The household problem in period t is as follows. In the morning, the household decides how much to consume and save, $c^*_{1,t}(\pi^{t-1})$ and $k^*_t(\pi^{t-1})$, under the expectation that the current period government action is $X_t(\pi^{t-1})$. At noon, π_t is revealed and the household decides on l and c_2. Call these decisions, $l_t = l(\pi_t)$ and $c_2(\pi_t)$. At time t, the (one-period-lived) government's problem is as follows. It selects an action, π_t, that optimizes discounted utility from time t on, subject to satisfying the time t budget constraint, and subject to the assumption that future government actions satisfy $X_{t+j}(\pi^{t+j-1})$, where π^{t+j} are the histories induced by π^{t-1}, π_t and X_{t+j}, $j = 1, 2, 3....$

A sustainable equilibrium is a sequence of actions, π_t, $t = 0, 1, 2, ...$, a sequence of functions, X_t, $c^*_{1,t}(\pi^{t-1})$ and $k^*_t(\pi^{t-1})$, $t = 0, 1, 2, ...$, and two functions $l(\pi)$, $c_2(\pi)$ satisfying the following characteristics:

(a) For all π^{t-1}, π_t, $c^*_{1,t}(\pi^{t-1})$, $k^*_t(\pi^{t-1})$, and $l(\pi_t)$, $c_2(\pi_t)$ satisfy the household problem. (Note, this does not just cover π^{t-1}, π_t such that $\pi_t = X_t(\pi^{t-1})$.)

(b) For all π^{t-1}, $\pi_t = X_t(\pi^{t-1})$ solves the government problem. (Note: this does not just cover histories, π^{t-1}, induced by X).

An *outcome* of a sustainable equilibrium is a sequence, $\{l_t, c_{2,t}, c_{1,t}, \pi_t\}$ induced by X_t in a sustainable equilibrium. The sequence of Ramsey outcomes can be the outcome of some sustainable equilibrium if β is sufficiently close to unity. This is a special case of the Folk Theorem. One could verify this as follows. Conjecture that the following objects form a sustainable equilibrium. $X_0 = (\delta^R, \tau^R)$, where δ^R, τ^R are the capital and labor tax rates that occur in a Ramsey equilibrium. Let $X_t(\pi^{t-1})$ be δ^R, τ^R for all π^{t-1} in which $\pi_t = (\delta^R, \tau^R)$ for all t. Let $X_t(\pi^{t-1})$ be δ^d, τ^d in all histories in which there was at least one deviation from (δ^R, τ^R). Here, δ^d, τ^d are the capital tax rate and labor tax rate in the one period sustainable equilibrium, when saving is zero. The key thing to verifying whether this is indeed a sustainable equilibrium requires establishing that for *every* history, even histories in which a past government has deviated, it is in the interest of the government to implement $X_t(\pi^{t-1})$. So, consider a history in which there has been a deviation, so that households expect (δ^d, τ^d). The current government has no better policy than to implement these tax rates because saving will be zero in any case, and then by the afternoon of that day, the government will be forced to implement τ^d to balance the budget. Now consider a history in which there has been no deviation. Now a deviation from $X_t(\pi^{t-1})$ triggers a move, starting in the next period, to the bad equilibrium. If β is sufficiently large, then the 'pain' felt by the government for this is extremely large. For example, if β is almost 1, it suffers 'pain' for an eternity in exchange for the one period gain it receives from implementing a huge capital levy and setting labor tax rates to zero. An infinite pain obviously always has to be worse than any finite gain, and so the government is not expected to deviate, if β is close enough to unity. Once this argument is formally established, then a sustainable equilibrium would have been identified which has the property that the corresponding outcome is Ramsey. It is perhaps obvious that if this equilibrium has Ramsey as an outcome, then it is likely that other equilibria with less wild expectations functions can perhaps also be found in which Ramsey is an outcome.