Homework #3
Economics D11-1, Fall 1998
Due Wednesday, October 14
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1. This question is designed to illustrate Blackwell's Theorem, Theorem 3.3 on page 54 of S-L. That theorem represents a set of conditions that are *sufficient* for a mapping, T, to be a contraction, so that $T^j w_0 = w$ as $j \to \infty$ for all w_0 belonging to a specified set. The question draws attention to the fact that the conditions of Blackwell's theorem are not *necessary.*

   Consider the following functional equation:

   $$T(v) = \max_{0 \leq \lambda \leq A+1-\delta} \frac{[A+1-\delta-\lambda]^{(1-\sigma)}}{1-\sigma} + \beta\lambda^{(1-\sigma)}v.$$

   Suppose $\sigma > 1$ and $\beta(A+1-\delta)^{1-\sigma} < 1$.

   (a) Show: $T(v) = \infty$ for $v > 0$, $T(0) = \frac{[A+1-\delta]^{(1-\sigma)}}{1-\sigma}$.

   (b) Show: the derivative of T at $v = v_0 < 0$ is:

   $$\frac{dT(v_0)}{dv} = \beta\lambda(v_0)^{(1-\sigma)},$$

   where

   $$\lambda(v_0) = argmax_{0 \leq \lambda \leq A+1-\delta} \frac{[A+1-\delta-\lambda]^{(1-\sigma)}}{1-\sigma} + \beta\lambda^{(1-\sigma)}v_0.$$

   (c) Explain why T does not satisfy the conditions of Theorem 3.3 in S-L, page 54. (Hint: does $T : B(X) \to B(X)$, where the 'functions' we consider here are actually points in R? Is discounting satisfied?)

   (d) What happens to $\lambda(v)$ as $v \to -\infty$?

   (e) What does the graph of $T(v)$ versus v for $v \leq 0$ look like? Does it cross a 45^0 line drawn in the negative orthant? Draw this graph by hand, emphasizing its qualitative features (i.e., you need not compute the graph numerically, using numerical values for the parameters of the function.)

(f) Explain, using the graph you just developed, why $T^j v_0 = v^*$ as $j \rightarrow \infty$, for every $v_0 < 0$, where v^* is unique.

2. Suppose a planner chooses to maximize, by choice of $c_0, c_1, c_2, \ldots$, the following expression:

$$u(c_0) + \delta[\beta u(c_1) + \beta^2 u(c_2) + \ldots], u(c_t) = \log(c_t)$$

subject to

$$c_t = k_t^\alpha - k_{t+1}, 0 < \alpha < 1, c_t, k_{t+1} \geq 0,\ k_0 \text{ given},$$

where $0 < \delta < \beta < 1$. When $\delta = 1$, this is the problem studied in exercises 2.2 and 4.9 in SL.

(a) Use the fact that $k' = \beta\alpha k^\alpha$ solves the version of the problem with $\delta = 1$ to establish that the solution to the problem with $\delta \neq 1$ has the form:

$$k_1 = g k_0^\alpha, k_{t+1} = \beta\alpha k_t^\alpha, t = 1, 2, \ldots,$$

where g is a scalar. Derive an explicit formula relating g to the parameters of the model, β, α, δ.

(b) Is there a unique k^* with the property $k_t \rightarrow k^*$ as $t \rightarrow \infty$ for all k_0? Display a formula relating k^* to the parameters of the model.

(c) Suppose $\beta = 1/1.03, \alpha = .36, \delta = .8$. Suppose $k_0 = k^*$. Display the values of $k_0, k_1, k_2, k_3, k_4, k_5$ that solve the problem as of date zero.

(d) Now suppose that when date 1 happens, the planner decides to reoptimize with respect to $k_2, k_3, \ldots$. The initial condition for this problem is k_1, the decision implemented by the planner last period. The planner's preferences over $c_t, t \geq 1$ are as follows:

$$u(c_1) + \delta[\beta u(c_2) + \beta^2 u(c_3) + \ldots]$$

and the resource constraint is as before. What values will the planner choose for k_1, k_2, k_3, k_4, k_5? If the planner chooses to reoptimize in this way every period, to what value will k_t actually tend?

(e) Are the values for k_2, k_3, k_4, k_5 chosen by the planner in date 1 the same as the values for these variables chosen in date 0? Why not? Because the chosen values for these variables differs between time 0 and time 1, this problem is said to be *time inconsistent.* If δ had been set to one, we would not have had this problem. Why not?

(f) Basically, the attitude of the planner is 'I'm very impatient today (the discount rate from period 0 to period 1 is $\beta\delta$), but I'll be less impatient tomorrow (the discount rate from period 1 to period 2 is β), so I'll consume a lot today and save a lot tomorrow.' Such an attitude is not time consistent because when tomorrow rolls around the planner says the same thing. In the end, the planner just ends up with a low capital stock. This type of model has been used to explain the behavior of smokers, who resolve that 'tomorrow I'll quit smoking, but tonight I'll just have one or two more'. Does the solution concept that we have used make any sense? Would a rational person make decisions in the time-inconsistent way described in (d) and (e), or would they do something else? Answers to this question often involve posing the problem as a game between the planner in period t and the planner in period $t+1$, and takes us beyond the scope of this course.

3. Consider the following preferences:

$$\sum_{t=0}^{\infty} \beta^t \{\log(c_t) + \gamma \log(c_{t-1})\}, \; 0 < \beta < 1, \; \gamma > 0.$$

Let the resource constraint be $c_t + k_{t+1} \leq A k_t^{\alpha}$, $A > 0$, $0 < \alpha < 1$, and suppose $k_0 > 0$, $c_{-1} > 0$ are given. Here, c_t is consumption at time t, and k_t is the capital stock at the beginning of period t. The form of the current utility function $\log(c_t) + \gamma \log(c_{t-1})$, captures the notion that consumption in one period may generate utility for more than one period.

(a) Let $v(k_0, c_{-1})$ be the value of $\sum_{t=0}^{\infty} \beta^t \{\log(c_t) + \gamma \log(c_{t-1})\}$ for a consumer who begins time 0 with capital stock k_0 and lagged consumption, c_{-1}, and behaves optimally. Display the functional equation for which v is a fixed point.

(b) Consider the following 'guess' of v_0 : $v_0(k, c_{-1}) = E_0 + F_0 \log(k) + G_0 \log(c_{-1})$. Define the T operator and consider $v_i = Tv_{i-1}$, $i = 1, 2, 3, \ldots$. Do the v_i's converge? What do they converge to? Give explicit formulas in terms of the parameters A, β, α, and γ.