Homework #4
Economics D11-1, Fall 1998
Due Saturday, October 24.
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The following two questions work with material covered in the course up to the end of last week. It may be useful for understanding the material covered by the midterm on October 21.

1. In the previous homework, there was a question about a sequence problem in which the objective is

$$u(c_0) + \delta[\beta u(c_1) + \beta^2 u(c_2) + \ldots], u(c_t) = \log(c_t), \tag{1}$$

and the constraints are:

$$c_t = k_t^\alpha - k_{t+1}, 0 < \alpha < 1, c_t, k_{t+1} \geq 0, \; k_0 \text{ given},$$

where $0 < \beta < 1$, and $0 < \delta \leq 1$.

You showed that if a central planner sat down at date 0 and made savings decisions for all time, with full ability to commit to the future decisions, the optimal policy has the form:

$$k_1 = g k_0^\alpha, k_{t+1} = \beta\alpha k_t^\alpha, t = 1, 2, \ldots,$$

where

$$g = \frac{\alpha\delta\beta}{1 - \alpha\beta(1-\delta)}, \tag{2}$$

so that $g < \beta\alpha$ when $\delta < 1$. This is the solution under *full commitment*. Now suppose the planner has no ability to commit to future policies. This is the case of *no commitment*. In this case, the planner at date 0 can be thought of as being a different planner than the one in date 1, who in turn is different from the one in date 2, etc. These planners resemble the type of household that says, *at each date*, 'today we spend freely on restaurants, movies and other fun things, and tomorrow we start saving'. It is clear that such a household has 'no commitment technology': it's as if the household at each date is different from all subsequent households, and none has any control over the others. The situation can be thought of as a game in which the players are the household at date 0, the household at date 1, etc. In the no commitment case, it is useful to think of the planner's problem in the same way.

In recitation, the notion of a Markov equilibrium in the game under no commitment was discussed. In the equilibrium, the date t planner solves the following problem: choose k_{t+1} as a fraction of date t income, k_t^α, to maximize utility from date t on, subject to the certain belief that: (i) all future planners save $100\tilde{g}$ percent of income and (ii) $\tilde{g}$ is beyond the control of the date t planner. This is the problem of each planner, for $t = 0, 1, 2, ...$. The fact that the date t planner cares about utility from t on, and believes that in future periods the saving rate will be $\tilde{g}$ implies that its problem has the form:

$$\max_{0 \leq k_{t+1} \leq k_t^\alpha} \left\{ \log(k_t^\alpha - k_{t+1}) + d + \frac{\delta\beta\alpha}{1-\beta\alpha} \log(k_{t+1}) \right\},$$

where d is a constant that is a function of $\tilde{g}$. The solution to this problem takes the from $k_{t+1} = \hat{g}k_t^\alpha$, where $\hat{g}$ is not a function of k_t. In general, you can think of problems like this as inducing a mapping from future decisions into present decisions: $\hat{g} = H(\tilde{g})$. In this special case, the mapping is trivial, in that H does not vary with $\tilde{g}$. In particular, $\hat{g}$ is (2) regardless of $\tilde{g}$. Thus, the fixed point of H is (2), as derived in recitation.

(a) An alternative way to compute the current period saving rate is to substitute the resource constraint into (1):

$$\log\left(k_0^\alpha - k_1\right) + \delta\left[\beta \log\left(k_1^\alpha - k_2\right) + \beta^2 \log\left(k_2^\alpha - k_3\right) + ...\right],$$

set the derivative with respect to k_1 to zero:

$$\frac{1}{k_0^\alpha - k_1} = \frac{\delta\beta\alpha k_1^{\alpha-1}}{k_1^\alpha - k_2},$$

and then impose $k_2 = \tilde{g}k_1^\alpha$. This also induces a mapping from $\tilde{g}$ to the current growth rate, $\tilde{H}(\tilde{g})$:

$$\tilde{H}(\tilde{g}) = \frac{\alpha\beta\delta}{1 - \tilde{g} + \alpha\beta\delta}.$$

The fixed point of this is $\tilde{g} = \alpha\beta\delta$, which is smaller than (2). Why is this so? How is the problem solved here different from the problem solved in the Markov equilibrium without commitment?

(b) Laibson, of Harvard, argues that real-world household preferences resemble the preferences in this question. He argues that societies deal with the resulting tendency to party too much and save too little by designing institutions that impose limited commitment. An example may be a rule that says that when you decide how much money to have automatically deducted from your paycheck for savings, you can only change that decision once per year, on January 1. We now consider a Markov equilibrium under this type of limited commitment.

Suppose that the date t planner takes k_{t+1}/k_t^α as given and chooses $g = k_{t+2}/k_{t+1}^\alpha$. The planner takes as given and beyond control the decisions of future planners, k_{t+j+1}/k_{t+j}^α, for $j = 2, 3, \ldots$.

i. Define a Markov equilibrium concept for this economy.

ii. Show that the saving rate in such an equilibrium is $g = \alpha\beta$.

iii. Imagine that each period is divided into two halves: the morning and the afternoon. In the no commitment Markov equilibrium, a given period's saving rate is decided and implemented in the morning. Suppose we are in the steady state of the no commitment Markov equilibrium and consider some particular date, date 0. At noon on that date, without prior debate or discussion, a constitutional amendment is suddenly implemented which imposes the following impossible-to-change rule. (Somehow, the society managed to acquire a commitment technology that makes this type of commitment feasible.) The new rule is that henceforth, beginning that very afternoon, savings decisions to be implemented on the following morning must be decided in the afternoon before.

A. How do the sequence of saving decisions implemented in date 0, 1, 2, compare with the sequence that a planner with full commitment would implement from date 0 on?

B. Suppose $\beta = 1/1.03, \alpha = .36, \delta = .8$. What is the *value* of the commitment technology? For this, compute present discounted utility from date 0 on, v^{nc}, assuming the society remains in the no commitment Markov equilibrium. Then, compute the discounted value of utility from date 0 on, v^{lc}, associated with the policy change. The difference in utility levels, $v^{lc} - n^{nc}$, is hard to interpret directly. A standard way to perform this comparison is to ask what is the maximum that people would be willing to pay, as a permanent and constant tax rate on consumption, to acquire the commitment technology. That is, compute τ such that

$$\sum_{t=0}^{\infty} \beta^t \log\left[c_t^{lc}(1-\tau)\right] = v^{nc},$$

where c_t^{lc} is the consumption enjoyed by households in the limited commitment equilibrium. What is the value of τ? (Hint: note that the left side of the equality can be written $log(1-\tau)/(1-\beta) + v^{lc}$.)

2. Consider an economy with a large number of identical households, each having preferences, $\sum_{t=0}^{\infty} \beta^t u(c_t)$. Suppose the resource constraint is $c_t + i_t \leq f(k_t) + (1-\delta)k$, where $k_{t+1} = i_t + (1-\delta)k_t$, f is strictly increasing and concave, $f'(k) \to \infty$ as $k \to 0$, $f'(k) \to 0$ as $k \to \infty$, $0 < \delta < 1$. Assume investment, i_t, is irreversible, i.e., it must be that $i_t \geq 0$. In addition, suppose $c_t, k_t \geq 0$ and that $k_0 > 0$ is given. Consider the functional equation associated with this problem:

$$\begin{aligned} v(k) &= \max_{k' \in \Gamma(k)} u\left(f(k) + (1-\delta)k - k'\right) + \beta v(k') \\ \Gamma(k) &= \{k' : (1-\delta)k \leq k' \leq f(k) + (1-\delta)k\}. \end{aligned}$$

(a) State a set of assumptions on β and u that guarantee there is a unique, differentiable, concave v that solves the above functional equation. For each property of v, explain which assumptions are used to get it.

(b) Show that monotonicity of $\Gamma(k)$, Assumption 4.6 in S-L, fails so that one of the conditions of Theorem 4.7 which guarantee strictly increasing v, is not satisfied.

(c) Show that the feasible set for this economy satisfies the following 'quasi-monotonicity property': if $\tilde{k} \geq k$, then $\Gamma(k) + (1-\delta)(\tilde{k} - k) \subseteq \Gamma(\tilde{k})$. Here, the sum of a set, say X, and a number, say a, is a new set, $X + a$, where $X + a \equiv \{x + a : x \in X\}$.

(d) Show: v is an increasing function in k. (Hint: (i) following the basic strategy of the proof of Theorem 4.7, it's enough to establish that the assumptions of Theorem 4.7 with the monotonicity assumption on Γ replaced by quasi-monotonicity guarantee Tw is increasing if w is; (ii) make use of the fact that if $k' \in \Gamma(k)$, then $\tilde{k}' = k' + (1-\delta)(\tilde{k} - k) \in \Gamma(\tilde{k})$, $\tilde{k}' > k'$, and $f(\tilde{k}) + (1-\delta)\tilde{k} - \tilde{k}' > f(k) + (1-\delta)k - k'$.) Can you provide intuition for the fact that v is increasing even though Γ fails to satisfy monotonicity?