

Homework #5
 Economics D11-1, Fall 1998
 Due Friday, November 30.
 Christiano

1. Consider an economy with capital of different vintages. At time t , the amount of capital of vintage τ , $k_{t,\tau}$, $\tau = 1, 2, 3, \dots$, is

$$k_{t,\tau} = \gamma^{t-\tau}(1 - \delta)^{\tau-1}i_{t-\tau},$$

where $\gamma > 1$, $0 < \delta < 1$, $i_{t-\tau}$ is the amount of investment, in time $t - \tau$ consumption units, applied in period $t - \tau$. Capital which has vintage τ in period t has vintage $\tau + 1$ in period $t + 1$. Investment expenditures at time t , i_t , must all be applied to the latest vintage (for a model in which investment in old vintages is feasible and desirable, see Chari and Hopenhayn, JPE, 1991) and results in $k_{t+1,1} = \gamma^t i_t$ units of new-vintage period $t + 1$ installed capital goods. Consider a given amount of investment, i . Note that this investment applied in period $t + 1$ produces more new-vintage installed capital (i.e., $\gamma^{t+1}i$) than the same level of investment applied in period t (i.e., $\gamma^t i$). This reflects the assumption, $\gamma > 1$ which is designed to capture the notion that there is exogenous technical progress that is embodied in new capital equipment. Note that the efficiency of a particular vintage stays constant over time, it's just that the efficiency of each succeeding vintage is greater than the efficiency of the previous one.

Capital of each vintage is operated with labor to produce a homogeneous output good, $y_{t,\tau}$ according to the following production function:

$$y_{t,\tau} = k_{t,\tau}^\alpha n_{t,\tau}^{1-\alpha}, \quad 0 < \alpha < 1, \quad \tau = 1, 2, 3, \dots$$

Suppose there is a competitive market in capital of different vintages and in labor. Each vintage of capital has the same rental rate, r_t , since capital is measured in common efficiency units. Similarly, the wage rate is w_t .

- (a) Show that a firm's profit maximizing choice of $n_{t,\tau}$ gives rise to the following relationships:

$$y_t = k_t^\alpha n_t^{1-\alpha}, \quad (1 - \alpha) \left(\frac{k_t}{n_t} \right)^\alpha = w_t, \quad \alpha \left(\frac{k_t}{n_t} \right)^{\alpha-1} = r_t,$$

where

$$y_t = \sum_{\tau=1}^{\infty} y_{t,\tau}, \quad k_t = \sum_{\tau=1}^{\infty} k_{t,\tau}, \quad n_t = \sum_{\tau=1}^{\infty} n_{t,\tau}.$$

(Hint: refer to your class notes on the indeterminacy of firm size under constant returns to scale.)

(b) Show that ‘aggregate capital’, k_t , evolves according to:

$$k_{t+1} = (1 - \delta)k_t + \gamma^t i_t.$$

2. Consider the two-sector economy, in which consumption and new capital are produced according to the following technologies,

$$c_t = k_{ct}^{\alpha} n_{ct}^{1-\alpha}, \quad k_{t+1} - (1 - \delta)k_t = z_t k_{it}^{\alpha} n_{it}^{1-\alpha},$$

respectively. The price, in units of date t consumption goods, of new capital goods is p_t . Firms in the two sectors are competitive in output, capital and labor markets and take the rental rate on capital, r_t , and the wage rate, w_t , as given. The investment good firm takes p_t as given. All these prices are denominated in units of the date t consumption good. Profits of the consumption good firms, denominated in consumption units, are $k_{ct}^{\alpha} n_{ct}^{1-\alpha} - r_t k_{ct} - w_t n_{ct}$, and profits of the investment good firms is $p_t z_t k_{it}^{\alpha} n_{it}^{1-\alpha} - r_t k_{it} - w_t n_{it}$.

(a) What is the restriction across r_t and w_t that must be satisfied in equilibrium? Assume from here on that that restriction is satisfied.

(b) Show that the capital-labor ratios in the two sectors are the same, i.e., $k_{ct}/n_{ct} = k_{it}/n_{it}$.

(c) Show that

$$c_t + i_t = k_t^{\alpha} n_t^{1-\alpha}, \quad \text{where } k_{t+1} - (1 - \delta)k_t = z_t i_t,$$

and

$$k_t = k_{ct} + k_{it}, \quad n_t = n_{ct} + n_{it},$$

and

$$\alpha \left(\frac{k_t}{n_t} \right)^{\alpha-1} = w_t, \quad (1 - \alpha) \left(\frac{k_t}{n_t} \right)^{\alpha} = r_t.$$