

Homework #6
Economics D11-1, Fall 1998
Due Friday, November 6.
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1. The following three sector exogenous growth model was recently proposed in Kongsamut, Rebelo and Xie ('Beyond Balanced Growth', National Bureau of Economic Research Working Paper number 6159, available on the web at <http://www.nber.org/>), to explain several key features of long-run (i.e., 1869-1990) growth: (i) K/Y is roughly constant, where K denotes the aggregate stock of capital, and Y denotes aggregate output, (ii) K grows at a roughly constant rate, (iii) the rate of return on capital is relatively constant, and (iv) resources have been reallocated out of agriculture and into services, while manufacturing has a relatively stable share in the economy.

Consider the following technology. Agricultural output, A_t , is produced using the following production function:

$$A_t = B_A K_{At}^\alpha (N_{At} z_t)^{1-\alpha},$$

where $B_A > 0$, $0 < \alpha < 1$, z_t denotes the state of technology, and K_A and N_A denote capital and labor allocated to agriculture. The state of technology evolves according:

$$z_t = \exp(g) z_{t-1}, \quad g > 0.$$

The manufacturing sector produces output that can be converted into consumption goods, C_t , or capital goods:

$$C_t + K_{t+1} - (1 - \delta)K_t = K_{Mt}^\alpha (N_{Mt} z_t)^{1-\alpha},$$

in obvious notation. Finally, services, S_t , are produced according to:

$$S_t = B_S K_{St}^\alpha (N_{St} z_t)^{1-\alpha}.$$

Suppose that at a point in time, households supply K_t units of capital to the capital rental market and 1 unit of labor to the labor markets, so that clearing in these markets requires:

$$N_{At} + N_{Bt} + N_{Mt} = 1, \quad K_{At} + K_{Bt} + K_{Mt} = K_t.$$

Prices are denominated in units of manufactured goods, so that the price of a manufactured good is unity. Let the price, in units of manufactured goods, of an agricultural good, be P_{At} . Let the price of a service be P_{St} . Finally, let r_t and w_t denote the rental rate and wage rate. Suppose that the three technologies are operated by competitive firms.

- (a) Show that competitive behavior by firms implies: $K_{At}/N_{At} = K_{Mt}/N_{Mt} = K_{St}/N_{St} = K_t$, $P_{At} = 1/B_A$, $P_{St} = 1/B_S$,

$$C_t + K_{t+1} - (1 - \delta)K_t + \frac{A_t}{B_A} + \frac{S_t}{B_S} = K_t^\alpha z_t^{1-\alpha}. \quad (1)$$

One can treat the latter as a ‘reduced form’ expression for the economy’s resource constraint.

- (b) Derive an expression for the rate of return on capital. If K_t grows at the rate, g , will the rate of return on capital be constant?
- (c) Suppose preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \frac{[(A_t - \bar{A})^\eta C_t^\gamma (S_t + \bar{S})^\theta]^{1-\sigma}}{1 - \sigma}, \quad \eta + \gamma + \theta = 1, \quad \eta, \gamma, \theta, \sigma > 0.$$

- i. Write out a budget constraint for households and define a sequence of markets equilibrium for this economy.
- ii. Derive the household’s intertemporal Euler equation associated with capital.
- iii. Show that household optimization, together with the results for prices you derived above, imply:

$$\frac{\gamma(A_t - \bar{A})}{\beta C_t} = B_A, \quad \frac{\gamma(S_t + \bar{S})}{\theta C_t} = B_S. \quad (2)$$

- iv. Substitute out for $A_t - \bar{A}$ and $S_t + \bar{S}$ in terms of C_t in the household’s intertemporal Euler equation, to get an expression in terms of the growth rate of C_t and the rate of return on capital alone.

- v. Show that in general, there is no reason to expect the economy to converge to a ‘balanced growth path’, i.e., one in which A_t, C, S_t, K_t, Y_t all grow at a constant rate. (Here, $Y_t = K_t^\alpha z_t^{1-\alpha}$.) Hint: note that if you scale A_t, M_t, K_t, S_t by z_t and work with the scaled versions of (2) and (1) and the household’s intertemporal Euler equation, you cannot get rid of z_t .
- vi. Consider the following restriction:

$$\bar{A}B_S = \bar{S}B_A. \quad (3)$$

Show that in this case, the economy boils down to the one sector growth model with disembodied technical change. Hint: note that in this case, you can replace A_t and S_t in (1) by $A_t - \bar{A}$ and $S_t - \bar{S}$, respectively without changing (1). Then, define $a_t = A_t - \bar{A}$ and $s_t = S_t - \bar{S}$ everywhere and impose (2), that a_t and s_t are each proportional to M_t .

- vii. Explain why it is that under (3), the following are true:

$$\begin{aligned} \frac{K_t}{z_t} &\rightarrow k^*, \quad \frac{A_t - \bar{A}}{z_t} \rightarrow a^*, \quad \frac{S_t - \bar{S}}{z_t} \rightarrow s^* \\ \frac{K_{t+1}}{K_t}, \frac{Y_{t+1}}{Y_t} &\rightarrow \exp(g), \end{aligned}$$

where k^*, a^*, s^* are finite, positive, constants.

- viii. Provide a simple formula for k^* .
 - ix. What happens to the rate of return on capital along a growth path? What happens to the distribution of employment between sectors along a growth path?
 - x. What does the model imply for P_{k^t} , the consumption price of capital, along the growth path.
2. Until now we have never worried about how investment is financed. That’s because we have always considered decentralizations in which the household buys additions to capital using current income, so there is no need for ‘finance’. This way of organizing things makes it impossible for us to think about an asset market variable like the rate of return

on equity. In this question we consider a decentralization in which the accumulation of capital is financed by entrepreneurs who put the capital to work with hired labor to produce output. The entrepreneurs issue equity and debt to finance their acquisition of capital. As a result, the environment in this question facilitates thinking about the equilibrium rate of return on equity and on other assets such as a one-period-ahead sure loan (our version of corporate debt).

The environment below has three properties: First, the equilibrium consumption, labor and capital stock quantities in this model are not dependent on the debt-to-equity ratio of the firm. This is a version of the celebrated Modigliani-Miller theorem in finance. Second, the equilibrium rate of return on equity does depend on the debt-to-equity ratio. This is because equity has to absorb all the uncertainty in the firm's revenue stream, and that is riskier as the firm is more leveraged with debt. The premium on the rate of return on equity over debt increases as the debt to equity ratio increases. Third, for this model to account for the empirically observed equity premium requires an implausibly high debt-to-equity ratio (this is consistent with the findings of a celebrated paper, Mehra and Prescott 'The Equity Premium: A Puzzle,' *Journal of Monetary Economics*, 1985).

Consider the following economy with households and firms.

Households

We suppose there are many identical households. The typical household takes prices, wages and rates of return as given. In a sequence-of-markets competitive environment, the household seeks at time t to maximize expected utility:

$$E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}) \quad (4)$$

subject to the sequence of budget constraints,

$$c_{t+j} + z_{t+j+1} + b_{t+j+1} = w_{t+j}n + R_{t+j}b_{t+j} + r_{t+j}z_{t+j}, \quad (5)$$

$j \geq 0$, and subject to initial levels of the stock of debt and equity, b_t and z_t . Here, c_t denotes consumption and household supply of labor, n ,

is assumed to be fixed. Equity and debt acquired in period t , z_{t+1} and b_{t+1} , have rates of return r_{t+1} and R_{t+1} , respectively. The return, r_{t+1} , is a function of date $t + 1$ (and possibly earlier) economic shocks, while R_{t+1} is only a function of shocks dated t and earlier. Thus, from the perspective of time t , the time $t + 1$ rate of return on debt is constant across date $t + 1$ states of nature, while the rate of return on r_{t+1} varies across date $t + 1$ states of nature. This feature of the return on debt leads us to refer to it as ‘conditionally sure’. The household takes r_t, w_t (wages) and R_t as given and beyond its control.

- (a) Write out the first order necessary conditions associated with the household’s choice of z_{t+1} and b_{t+1} .
- (b) Define the date t equity premium, P_t , to be the excess of the date t conditionally expected return on equity, $E_t r_{t+1}$, over the conditional sure return on bonds, R_{t+1} :

$$P_t = \frac{E_t r_{t+1}}{R_{t+1}}.$$

- (c) Show that

$$P_t = 1 - Cov_t(m_{t+1}, r_{t+1}),$$

where $m_{t+1} = \beta u_{c,t+1}/u_{c,t}$, is the intertemporal marginal rate of substitution in consumption and $u_{c,t}$ is the date t marginal utility of consumption. (Hint: use the fact, $Cov_t(x_{t+1}, y_{t+1}) = E_t x_{t+1} y_{t+1} - E_t x_{t+1} E_t y_{t+1}$, and the first order conditions developed in (a).)

- (d) Suppose the conditional covariance in the above expression were positive. Then P_t is less than 1, i.e., the conditionally expected rate of return on equity is less than that on debt. This seems peculiar. Why should people be willing to hold equity when its payoff is uncertain, and lower in expected value than debt? Explain in intuitive terms.

Firms

We suppose there are many firms, all of which are identical. The typical firm takes prices, w_t and rates of return, R_t, r_t , as given.

In period $t + 1$ the firm uses capital, K_{t+1} , and labor, N_{t+1} , to produce output, Y_{t+1} , using the following production function:

$$Y_{t+1} = F(K_{t+1}, N_{t+1}, \epsilon_{t+1}), \quad (6)$$

where F is linear homogeneous in its first two arguments and ϵ_{t+1} is a stationary random shock to technology. An entrepreneur who wishes to operate the firm in period $t + 1$ must during period t acquire the purchasing power needed to purchase K_{t+1} from the current firm. The entrepreneur acquires this purchasing power by selling equity shares, Z_{t+1} , and debt, B_{t+1} , to the household. The entrepreneur's period t financing constraint is:

$$Z_{t+1} + B_{t+1} = K_{t+1}. \quad (7)$$

Next period, the entrepreneur brings $Y_{t+1} + (1 - \delta)K_{t+1}$ to the goods market to sell. This includes new production, Y_{t+1} , and the capital stock that remains after depreciation at the end of next period, $(1 - \delta)K_{t+1}$. The entrepreneur's expenses next period include the wage bill, $w_{t+1}N_{t+1}$, and the obligations on debt, $R_{t+1}B_{t+1}$, and equity, $r_{t+1}Z_{t+1}$. Thus, the entrepreneur's total cash flow at the end of $t+1$ is π_{t+1} , where,

$$\pi_{t+1} = Y_{t+1} + (1 - \delta)K_{t+1} - w_{t+1}N_{t+1} - R_{t+1}B_{t+1} - r_{t+1}Z_{t+1} \geq 0, \quad (8)$$

where the inequality constraint reflects that the entrepreneur cannot pay out what he/she does not have (this is sometimes referred to as the 'no blood from stone' constraint). At date t , when the entrepreneur must choose Z_{t+1} and B_{t+1} , the date $t + 1$ cash flow is unknown, since the date $t + 1$ date of nature is not yet realized. In a sense, the entrepreneur is a multiproduct firm. By choosing values of Z_{t+1} and B_{t+1} multiple goods are produced - corresponding to the cash flow in each possible date $t + 1$ state of nature. The entrepreneur weighs the different π_{t+1} 's corresponding to different realizations of ϵ_{t+1} by the product of the probability of that realization and the associated marginal utility of consumption. We use this to define the entrepreneur's profit function.

- (e) Suppose that at date t there were markets for date $t + 1$ state-contingent goods. Show that the weights that we apply to profits across states of nature correspond to the prices that would obtain in such markets for state-contingent goods. Thus, the entrepreneur's profit function is the standard profit function for a multiproduct firm.
- (f) The entrepreneur's profit function is $E_t u_{c,t+1} \pi_{t+1}$, where $u_{c,t+1}$ is treated as exogenous by the entrepreneur. The entrepreneur solves

$$\max_{Z_{t+1}, B_{t+1}} E_t \{ u_{c,t+1} \max_{N_{t+1}} \pi_{t+1} \}. \quad (9)$$

The maximization inside the braces reflects that the firm's employment decision is made after the realization of ϵ_{t+1} . The maximization outside the braces reflects that the financing decision is made before the realization of ϵ_{t+1} . Make sure you understand this point.

- (g) Write out the first order necessary conditions associated with the firm's choice of Z_{t+1} , B_{t+1} , and N_{t+1} . Suppose that there is free entry, so that, in equilibrium, $E_t \{ u_{c,t+1} \pi_{t+1} \} = 0$. Explain why this implies that $\pi_t = 0$ in every state of nature.
- (h) Define a sequence of markets equilibrium and a recursive competitive equilibrium for this economy.
- (i) Explain why the debt-to-equity ratio, b_{t+1}/z_{t+1} , is not pinned down in equilibrium. In particular, if there is an equilibrium in which some particular value of $\gamma = b_{t+1}/z_{t+1}$ holds, then there exists an equilibrium with the same allocations of consumption, output and capital, for all other values of γ .
- (j) Show, using the linear homogeneity assumption on F , the firm's first order condition on labor, the entrepreneur's financing constraint, and the zero ex post profit condition, that

$$r_{t+1} = (F_{k,t+1} + 1 - \delta)(1 + \gamma) - \gamma R_{t+1}.$$

Use this to show that the equity premium, P_t , increases with γ .

- (k) Let $u(c_t) = \log(c_t)$ and resource constraint, $c_t + k_{t+1} = k_t^\alpha n_t^{1-\alpha} \exp(x_t)$, where x_t has a first order autoregressive representation: $x_t = \rho x_{t-1} + \varepsilon_t$, where ε_t iid over time and independent of x_{t-1} . Show that the equity premium, P_t , reduces to:

$$P_t = 1 - Cov_t(\exp(-\varepsilon_{t+1}), \exp(\varepsilon_{t+1}))(1 + \gamma),$$

where γ is the debt/equity ratio, assumed to be fixed.

Suppose ε_t is iid over time with $\varepsilon_t = \sigma$ with probability 1/2 and $\varepsilon_t = -\sigma$ with probability 1/2. It is easily verified that $Var(\varepsilon_t) = (\sigma)^2$, so that σ is the standard deviation of ε_t . Based on his analysis of quarterly U.S. data, Prescott (1986, Federal Reserve Bank of Minneapolis Quarterly Review) has argued that an empirically plausible value for this quantity is 0.00763. Compute P_t for the case $\gamma = 0$.

Mehra and Prescott argued that the equity premium in the U.S. averages 1.07 percent per annum, or, 1.017 per quarter. What value of γ is necessary to make P_t this large?