

Homework #8
 Economics D11-1
 Due Friday, November 20.
 Christiano

1. Consider the model economy associated with Romer's model of growth through specialization. That is, preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\gamma}}{1-\gamma}, \gamma > 0.$$

The technology for producing final goods is:

$$y_t = \int_0^{M_t} x_t(i)^\alpha di, \quad M_t > 0, \quad 0 < \alpha < 1,$$

where M_t is a scalar such that for $i > M_t$, $x_t(i) = 0$. To produce $x_t(i)$ units of the i^{th} intermediate good requires

$$\frac{1}{2}(1 + x_t(i)^2)$$

units of capital if $x_t(i) > 0$ and zero units of capital if $x_t(i) = 0$. The following constraint must be satisfied:

$$\int_0^{M_t} \frac{1}{2}(1 + x_t(i)^2) di = k_t,$$

where k_t is the beginning-of-period t aggregate stock of capital. The initial capital stock, $k_0 > 0$, is given. The resource constraint is:

$$c_t + I_t \leq y_t,$$

and the aggregate capital accumulation technology is given by:

$$k_{t+1} = (1 - \delta)k_t + I_t.$$

The efficient allocations for this economy solve the planning problem, maximize utility with respect to $\{M_t, k_{t+1}, y_t, c_t, x_t(i), i \in (0, M_t)\}_{t=0}^{\infty}$, subject to the various constraints. You may assume that efficiency is consistent with $x_t(i) = \bar{x}_t$ for $i \in (0, M_t)$.

- (a) Show that the planning problem for the Romer economy coincides with the planning problem for the Ak model. In particular, show that the problem can be written,

$$\max_{\{k_{t+1} \in B(k_t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1}),$$

where

$$F(k, k') = \max_{c_t, \bar{x}_t, M_t} \frac{c_t^{1-\gamma}}{1-\gamma} = \frac{[(A+1-\delta)k - k']^{1-\gamma}}{1-\gamma},$$

and $A = (2 - \alpha) \left(\frac{\alpha}{2 - \alpha} \right)^{\frac{\alpha}{2}}$. In addition to verifying the form of F , show what B is.

- (b) Identify a set of parameter values under which positive growth is efficient, although the growth rate in the market decentralization analyzed in class is zero.
- (c) The problem with monopoly power is that it results in an inefficiently low level of activity. In the Romer model we have just seen that this manifests itself in the form of inefficiently low growth. The pace at which new varieties of specialized inputs (e.g., specialized manufactured goods, specialized labor) are introduced is too slow in the market economy. Some sort of intervention in the market economy is desirable. One possibility is to subsidize the activities of monopolists. Accordingly, let $p(i)x(i)$ be the revenues of the i^{th} monopolist in the absence of taxes or subsidies. A subsidy rate, τ_t , raises the revenues of the i^{th} monopolists to $p(i)x(i)(1 + \tau_t)$. The total cost, G_t , to the government of this subsidy scheme is

$$G_t = \int_0^{M_t} p(i)x(i)\tau_t di.$$

Suppose G_t is financed by a lump sum tax applied to households. That is, the household budget constraint is modified as follows:

$$c_t + k_{t+1} - (1 - \delta)k_t = r_t k_t + w_t n_t - T_t,$$

where T_t represents taxes paid by the representative household to the government. Suppose the government balances its budget

period by period:

$$T_t = G_t.$$

Find the subsidy rate, τ_t , that causes the allocations in the market economy to coincide with the efficient allocations.

These results have to be interpreted with caution. You have identified an ideal form of government intervention, which makes the private market economy efficient. However, the intervention we investigated abstracts from any social inefficiencies induced by having to raise the revenues needed to finance the subsidy to monopolists. We abstracted from this by assuming that the tax on households is administered in lump-sum form. In practice, such taxes are not available. So, the problem of ‘fixing’ the inefficiency in the Romer model is actually more complicated than this question makes it out to be.

- (d) Devise a tax transfer scheme that subsidizes saving, which supports the efficient allocations.
2. Consider the Jones-Manuelli model discussed in class: Each period a continuum of identical, two-period lived agents is born. The typical period t agent has lifetime preferences:

$$\log(c_t^t) + \beta \log(c_{t+1}^t), \quad 0 < \beta < 1,$$

where c_j^t is the period j consumption of the typical agent born in period t . Each agent supplies one unit of labor inelastically in the first period of life, and none in the second (‘retirement’) period. Agents born in period t face the following sequence of budget constraints:

$$c_t^t + k_{t+1} \leq w_t, \quad c_{t+1}^t \leq (1 - \delta)k_{t+1} + r_{t+1}k_{t+1},$$

where w_t denotes the wage rate in period t , and r_{t+1} is the period $t + 1$ rental rate on capital. Also, k_{t+1} denotes the beginning of period $t + 1$ stock of capital held by the typical agent born in period t . Agents choose consumption and capital to maximize utility subject to non-negativity of consumption and capital.

Consumption and investment goods are produced by competitive firms using the following technology:

$$f(k, n) = b[\alpha k^\rho + (1 - \alpha)n^\rho]^{\frac{1}{\rho}}, \quad 0 < \rho < 1.$$

Goods producing firms maximize profits, $f(k, n) - wn - rk$.

The resource constraint in the production of goods is:

$$c_t^t + c_t^{t-1} + i_t \leq f(k_t, 1),$$

where ‘1’ denotes the amount of work effort supplied by the current young. The capital accumulation technology is:

$$k_{t+1} = (1 - \delta)k_t + i_t,$$

where $0 < \delta < 1$ is the rate of depreciation.

- (a) Show that there is no equilibrium with $k_{t+1}/k_t > 1$ for all t .
 - (b) Show that, although growth in the economic aggregates is not possible, there are parameter values for which growth in individual consumption nevertheless *is* possible. (Hint: construct a stationary equilibrium, in which all aggregates are constant through time, while $c_{t+1}^t/c_t^t = \lambda > 1$ for all t .)
 - (c) Devise a tax transfer scheme that will produce a positive growth rate in equilibrium for this economy.
3. (Matsuyama, forthcoming *Econometrica* article.) The model can be viewed as an extension of the Romer model analyzed above and discussed in class. We will use the same strategy as the one used in class, by first studying the partial equilibrium of the firm sector, and then going to general equilibrium by bringing in households. As before, since the partial equilibrium of the firm sector is static, in this part of the analysis we do not use time subscripts.

As in the Romer model, final good firms are competitive and make use of the linear homogeneous production function, $y = n^{1-\alpha} \int_0^M x(i)^\alpha di$, $0 < \alpha < 1$, where n is labor, $x(i)$ is the i^{th} intermediate input, and M is the variety of intermediate goods available. Final good firms take the wage rate, w , and the i^{th} intermediate good price, $p(i)$, as given and beyond their control. Profit maximization leads to first order conditions:

$$(1 - \alpha)y/n = w, \quad \alpha n^{1-\alpha} x(i)^{\alpha-1} = p(i), \quad 0 \leq i \leq M.$$

The differences between the Romer and the Matsuyama models center on the intermediate good firms. Matsuyama specifies that there are two types of intermediate good firms: those that produce old products, $x(i)$, $0 \leq i \leq M_{-1}$, and those that produce new products, $x(i)$, $M_{-1} < i \leq M$. Here, M_{-1} is the variety of goods in existence in the previous period. We suppose:

$$M \geq M_{-1}.$$

When $M = M_{-1}$, then there are no new goods.

Intermediate good firms producing old products are competitive (i.e., they take their output price as invariant to their own decisions) and firms producing new products (if there are any) produce monopolistically (i.e., they take into account the demand curve for their product). You can proceed as though there is just one representative competitive firm. Just make sure to take into account that it treats prices as parametric.

To produce $x(i)$ units of the intermediate good, $h(x(i); M_{-1})$ units of capital are needed, where

$$h(x(i); M_{-1}) = \begin{cases} x(i) + F, & \text{if } M_{-1} < i \leq M \text{ and } x(i) > 0 \\ 0, & \text{if } x(i) = 0, 0 \leq i \leq M \\ x(i), & 0 \leq i \leq M_{-1}. \end{cases}$$

Thus, to produce a new good requires paying a fixed cost, $F > 0$. We can think of the introduction of a new good as an innovation, so that F is the cost of innovating. The given specification of h indicates that the fixed cost only applies when a good is first introduced. As in the Romer model, monopoly power gives innovators the profits they need to cover the fixed costs of innovating.

We will consider a symmetric equilibrium, in which $x^c = x(i)$, $p^c = p(i)$ for $0 \leq i \leq M_{-1}$ and, *in case* $M > M_{-1}$, then $x^m = x(i)$, $p^m = p(i)$ for $M_{-1} < i \leq M$. Here, $x^m > 0$ is the profit maximizing level of output that a monopolist who ignores F would choose. The *actual* level of output that an innovator produces is either x^m or zero, whichever yields a higher level of profits.

Take M_{-1} , K , and n as given, where K denotes the capital stock. The

variables in the firm sector that are to be determined are

$$[p^m, x^m, p^c, x^c, y, w, r, M - M_{-1}].$$

(a) Show:

$$x^m = \left[\frac{\alpha^2}{r} \right]^{\frac{1}{1-\alpha}} n.$$

(b) Show:

$$p^m = \frac{r}{\alpha}, \quad p^c = r.$$

(c) Show, using the previous result and the demand curves for the intermediate inputs that,

$$\frac{x^c}{x^m} = \alpha^{\frac{1}{\alpha-1}}, \quad \frac{p^c x^c}{p^m x^m} = \alpha^{\frac{\alpha}{\alpha-1}} \equiv \theta > 1.$$

(d) Because there is free entry of monopolists, monopoly profits cannot be positive in equilibrium. If a potential monopolist expects to make negative profits by innovating, then $M = M_{-1}$, i.e., there will be no innovation. If $M > M_{-1}$, so that there is innovation, then monopoly profits are zero because of free entry. Explain why this corresponds to

$$x^m \leq \frac{\alpha}{1-\alpha} F, \quad \left[x^m - \frac{\alpha}{1-\alpha} F \right] [M - M_{-1}] = 0.$$

(e) Show that clearing in the capital market requires:

$$k = x^c + \frac{M - M_{-1}}{M_{-1}} (x^m + F).$$

where

$$k = \frac{K}{M_{-1}}.$$

(f) Show:

$$x^c = \alpha^{\frac{1}{\alpha-1}} x^m = \min \left\{ k, \frac{\theta}{1-\alpha} F \right\}.$$

(Hint: the first equality was shown previously. The second equality is based on the first equality and on the previous two results that you derived.)

(g) Show:

$$\frac{M - M_{-1}}{M_{-1}} = \begin{cases} 0, & \text{if } k < \frac{\theta F}{1-\alpha} \\ \frac{1-\alpha}{F} \left[k - \frac{\theta F}{1-\alpha} \right], & \text{if } k > \frac{\theta F}{1-\alpha} \end{cases}$$

(Hint: use the previous result, the condition $M \geq M_{-1}$, and the capital market clearing condition.) Thus, innovation (i.e., $M - M_{-1} > 0$) will occur only if there is a lot of physical capital relative to the amount of variety of goods (i.e., k is large). No innovation will occur otherwise. This makes sense because when capital is relatively abundant, we can expect its rental rate to be relatively low, increasing the incentive to innovate.

(h) Show:

$$\frac{y}{M_{-1}} = \begin{cases} n^{1-\alpha} k^\alpha, & \text{if } k < \frac{\theta F}{1-\alpha} \\ n^{1-\alpha} A k, & \text{if } k > \frac{\theta F}{1-\alpha} \end{cases}$$

where $A = (\theta F / (1 - \alpha))^{\alpha-1}$. Note how this aggregate production relation is continuous in k .

(i) Show:

$$r = \begin{cases} \alpha n^{1-\alpha} k^{\alpha-1}, & \text{if } k < \frac{\theta F}{1-\alpha} \\ \alpha n^{1-\alpha} A, & \text{if } k > \frac{\theta F}{1-\alpha} \end{cases} \quad (1)$$

By hand, draw a graph with r on the vertical axis and k on the horizontal axis. Let k vary from nearly zero to a value exceeding $\theta F / (1 - \alpha)$. Do the same for $\beta [r + 1 - \delta]$. You need only depict the qualitative behavior of these functions. Note how these functions are continuous in k .

(j) Suppose households supply $n_t = 1$ inelastically. Let their preferences be $\sum_{t=0}^{\infty} \beta^t \log(c_t)$, and let their budget constraint be

$$c_t + I_t \leq w_t + r_t K_t,$$

where $K_{t+1} = (1 - \delta)K_t + I_t$. The resource constraint for this economy is:

$$c_t + I_t \leq y_t.$$

Define a sequence of markets equilibrium.

- (k) For obvious reasons, Matsuyama refers to a state in which $k < \theta F/(1-\alpha)$ as a ‘Solow regime’ and a state in which $k > \theta F/(1-\alpha)$ as a ‘Romer regime’. Let

$$G = \beta \left[\alpha \left(\frac{\theta F}{1-\alpha} \right)^{\alpha-1} + 1 - \delta \right].$$

- i. Suppose $G < 1$. Show that there is a steady state value of k in the Solow regime, call it k^* . That is, for any $M_{-1} > 0$ if the initial stock of capital is $K_0 = k^*M_{-1}$, then there is a no growth equilibrium with

$$K_{t+1} = K_0, \text{ for } t = 0, 1, 2, \dots$$

Note that in this steady state equilibrium, there is never any innovation. This regime is more likely the larger is F , which makes sense because this represents the fixed cost of innovation.

Let $M_{-1} = 1$, $\beta = 1/1.03$, $\alpha = 0.36$, $F = 100$, so that $G = 0.8833$, after rounding. Compute k^* . If K_0 is in a sufficiently small neighborhood of k^*M_{-1} , show that there exists an equilibrium in which $\lim_{t \rightarrow \infty} K_t = k^*M_{-1}$. (Hint: consider the household’s intertemporal Euler equation after substituting out for the rental rate of capital using (1). Use the following facts: (i) for K_t sufficiently close to k^*M_{-1} , the Taylor series expansion of this equation about $K_t = K_{t+1} = K_{t+2} = k^*M_{-1}$ is an arbitrarily good approximation to this equation, and write this as

$$V_0 \tilde{K}_t + V_1 \tilde{K}_{t+1} + V_2 \tilde{K}_{t+2} = 0, \quad t = 0, 1, 2, \dots$$

where $\tilde{K}_t \equiv K_t - k^*M_{-1}$; (ii) the set of solutions to a linear difference equation like this is given by $\tilde{K}_t = (\tilde{K}_0 - a)\lambda_1^t + a\lambda_2^t$, for $t = 0, 1, 2, \dots$, for arbitrary a , where the λ_i ’s solve:

$$V_0 + V_1\lambda_i + V_2\lambda_i^2 = 0, \quad i = 1, 2;$$

(iii) an equilibrium must also satisfy a particular transversality condition.)

- ii. Suppose $G > 1$. Show that there is a steady state value of k in the Romer regime, call it k^{**} . That is, given $M_{-1} > 0$ and $K_0 > 0$, there is an equilibrium in which

$$\frac{K_t}{M_{t-1}} = k^{**}, \quad \frac{c_{t+1}}{c_t} = \frac{K_{t+1}}{K_t} = \frac{M_t}{M_{t-1}} = G, \quad t = 0, 1, 2, \dots$$

Provide a formula for computing k^{**} and verify $k^{**} > \theta F / (1 - \alpha)$.

- iii. Think about the possibility of equilibria that fluctuate between the Romer and Solow regimes: in a Solow regime the relatively low amount of physical capital results in a high rental rate on capital. This discourages innovation but encourages capital accumulation (just like in the neoclassical growth model when you are below steady state). When capital becomes relatively abundant (so that $k > \theta F / (1 - \alpha)$) then innovators have an incentive to enter: the Romer regime begins and M starts to grow. If M grows fast enough relative to K (this will depend upon parameter values) then k is driven back down towards the Solow regime, and the process starts all over again. Along such a growth path there will be alternating periods of fast growth during which there is no innovation and slow growth, during which there is a lot of innovation. Interestingly, the same conditions that encourage high growth in capital and output, i.e., a high rental rate of capital, discourage innovation. This model generates all sorts of empirical hypotheses that would be interesting to test (patent applications come in bursts, and at times of low growth?).