1. (35) Suppose agents in the economy have preferences \( P_t = 0^\infty u(c_t, n_t) \), \( 0 < \beta < 1 \). Here, \( u \) is twice differentiable, strictly concave, strictly increasing in \( c \) and decreasing in \( n \) for \( c > 0 \) and \( 0 < n < 1 \). The resource constraint is \( c_t + k_{t+1} \leq k_t^\alpha n_t^{1-\alpha} + (1 - \delta)k_t \), where \( 0 < \alpha < 1 \), \( 0 < \delta < 1 \). The constraints are \( c_t, n_t, k_{t+1} \geq 0 \) for \( t = 0, 1, 2, \ldots \). Also, \( k_0 \) is given at time 0.

(a) (7.5) Write this in the S-L canonical form:

\[
\max_{k_{t+1} \in \Gamma(k_t)} \sum_{t=0}^\infty \beta^t F(k_t, k_{t+1}).
\]

Explain carefully how to define \( F \) and \( \Gamma \).

(b) (10) Show that the function, \( F \), is strictly concave in its first argument. Explain in detail.

(c) (10) Show that the function, \( F \), is differentiable in its first argument. Explain in detail. Display a formula for the derivative of \( F \) in terms of the assumed preferences and technology.

(d) (7.5) Suppose \( u(c_t, n_t) = \log \left[ c_t - \frac{\psi_t}{1+\psi} n_t^{1+\psi} \right] \). Display the exact function, \( F(k, k') \), that holds in this economy for an interior equilibrium.

2. (20) Suppose the resource constraint has the form,

\[
c_t + g + k_{t+1} \leq f(k_t) + (1 - \delta)k_t, \quad 0 < \delta < 1,
\]

where \( f \) is increasing, \( f(0) = 0 \), \( f_k \to \infty \) as \( k \to 0 \) and \( f_k \to 0 \) as \( k \to \infty \), and \( g > 0 \) is government spending.
(a) (6.3) Define the constraint set, $\Gamma(k)$, in the S-L canonical form for this economy. Show that there is a always a lowest number, $k^{lb} > 0$, such that $\Gamma(k)$ is non-empty for all $k \geq k^{lb}$.

(b) (6.3) Show that, if $g$ is small enough, there is value for the capital stock, $k > k^{lb}$, with the following property. If $k^{lb} < k < k$ then the economy is not viable in the sense that the only feasible option is for $k$ to fall and eventually drop below $k^{lb}$.

(c) (6.3) Show that if $g$ is large enough, there is no value of $k$ such that the economy is viable.

(d) (.1) Suppose preferences have the form, $u(c)$. What is the optimal value of $g$?

3. (25) Consider a model in which utility is a function not just of the usual consumption, $c$, and labor effort, $l$, but also of a home-produced consumption good, $c_n$, and home labor effort, $l_n$. Specifically,

$$\log (c + c_n) - \gamma \log \left(\frac{l^{1+\psi}}{1+\psi} + l_n\right),$$

where $\gamma, \psi > 0$. The home labor effort yields services via the home production function, $c_n = \psi_0 l_n$. Show that this formulation implies a utility function in terms of $c$ and $l$ having the following form:

$$\text{constant} + a \log \left(c - \psi_0 \frac{l^{1+\psi}}{1+\psi}\right),$$

where ‘constant’ and $a$ are parameters. Display expressions for $a$ and $c$.

4. (20) Suppose we have a model economy in the S-L canonical form, and it satisfies all the usual assumptions (i.e., A4.3 – A4.9). Suppose further that there exist three distinct numbers, $x_1, x_2, x_3 \in R$, such that:

$$F_2(x_1, x_2) + \beta F_1(x_2, x_3) = 0,$$

$$F_2(x_2, x_3) + \beta F_1(x_3, x_1) = 0,$$

$$F_2(x_3, x_1) + \beta F_1(x_1, x_2) = 0,$$
and \( x_2 \in \text{int} [\Gamma(x_1)] \), \( x_3 \in \text{int} [\Gamma(x_2)] \), \( x_1 \in \text{int} [\Gamma(x_3)] \). Suppose the initial given stock of capital is \( k_0 = x_1 \). Display a sequence, \( k_2, k_3, k_4, k_5, \ldots \) which optimizes the sequence problem in the S-L canonical form. Justify your answer carefully.