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D11-1, Fall 1998

MIDTERM EXAM

There are four questions. The total number of possible points is 100, and the number of points per question is given in parentheses. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have 1 hour and 50 minutes. Good luck!

1. (35) Suppose agents in the economy have preferences $\sum_{t=0}^{\infty} \beta^t u(c_t, n_t)$, $0 < \beta < 1$. Here, u is twice differentiable, strictly concave, strictly increasing in c and decreasing in n for $c > 0$ and $0 < n < 1$. The resource constraint is $c_t + k_{t+1} \leq k_t^{\alpha} n_t^{1-\alpha} + (1-\delta)k_t$, where $0 < \alpha < 1$, $0 < \delta < 1$. The constraints are $c_t, n_t, k_{t+1} \geq 0$ for $t = 0, 1, 2, \ldots$. Also, k_0 is given at time 0.

   (a) (7.5) Write this in the S-L canonical form:

   $$\max_{k_{t+1} \in \Gamma(k_t)} \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1}).$$

   Explain carefully how to define F and Γ.

   (b) (10) Show that the function, F, is strictly concave in its first argument. Explain in detail.

   (c) (10) Show that the function, F, is differentiable in its first argument. Explain in detail. Display a formula for the derivative of F in terms of the assumed preferences and technology.

   (d) (7.5) Suppose $u(c_t, n_t) = \log\left[c_t - \frac{\psi_0}{1+\psi} n_t^{1+\psi}\right]$. Display the exact function, $F(k, k')$, that holds in this economy for an interior equilibrium.

2. (20) Suppose the resource constraint has the form,

   $$c_t + g + k_{t+1} \leq f(k_t) + (1-\delta)k_t, \; 0 < \delta < 1,$$

   where f is increasing, $f(0) = 0$, $f_k \to \infty$ as $k \to 0$ and $f_k \to 0$ as $k \to \infty$, and $g > 0$ is government spending.

(a) (6.3) Define the constraint set, $\Gamma(k)$, in the S-L canonical form for this economy. Show that there is a always a lowest number, $k^{lb} > 0$, such that $\Gamma(k)$ is non-empty for all $k \geq k^{lb}$.

(b) (6.3) Show that, if g is small enough, there is value for the capital stock, $\underline{k} > k^{lb}$, with the following property. If $k^{lb} < k < \underline{k}$ then the economy is *not viable* in the sense that the only feasible option is for k to fall and eventually drop below k^{lb}.

(c) (6.3) Show that if g is large enough, there is no value of k such that the economy is viable.

(d) (.1) Suppose preferences have the form, $u(c)$. What is the optimal value of g?

3. (25) Consider a model in which utility is a function not just of the usual consumption, c, and labor effort, l, but also of a home-produced consumption good, c_n, and home labor effort, l_n. Specifically,

$$\log\left(c + c_n\right) - \gamma \log\left(\frac{l^{1+\psi}}{1+\psi} + l_n\right),$$

where $\gamma, \psi > 0$. The home labor effort yields services via the home production function, $c_n = \psi_0 l_n$. Show that this formulation implies a utility function in terms of c and l having the following form:

$$\text{constant} + a \log\left(c - \psi_0 \frac{l^{1+\psi}}{1+\psi}\right),$$

where 'constant' and a are parameters. Display expressions for a and c.

4. (20) Suppose we have a model economy in the S-L canonical form, and it satisfies all the usual assumptions (i.e., $A4.3 - A4.9$). Suppose further that there exist three distinct numbers, $x_1, x_2, x_3 \in R$, such that:

$$\begin{aligned} F_2(x_1, x_2) + \beta F_1(x_2, x_3) &= 0, \\ F_2(x_2, x_3) + \beta F_1(x_3, x_1) &= 0, \\ F_2(x_3, x_1) + \beta F_1(x_1, x_2) &= 0, \end{aligned}$$

and $x_2 \in int\left[\Gamma(x_1)\right], x_3 \in int\left[\Gamma(x_2)\right], x_1 \in int\left[\Gamma(x_3)\right]$. Suppose the initial given stock of capital is $k_0 = x_1$. Display a sequence, $k_2, k_3, k_4, k_5, ...$ which optimizes the sequence problem in the S-L canonical form. Justify your answer carefully.