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FINAL EXAM

There are four questions. The total number of possible points is 100, and each question is worth the same number of points. If a question seems ambiguous, state why, sharpen it up and answer the revised question. You have 2 hours. Good luck!

1. Consider the following economy with a perfectly competitive final goods sector, and a monopolistically competitive intermediate goods sector. At time t the typical firm in the final goods sector produces output, y_t, from inputs, y_{it}, $i \in (0,1)$, using the linear homogenous production function:

$$y_t = \left[\int_0^1 y_{it}^{\lambda} di \right]^{\frac{1}{\lambda}}, 0 < \lambda < 1.$$

Profits of the final goods firms are given by:

$$p_t y_t - \int_0^1 p_{it} y_{it} di,$$

where p_{it} is the price of the i^{th} intermediate input, $i \in (0,1)$.

(a) Show that the demand function for the i^{th} intermediate good is given by:

$$y_{it} = y_t \left(\frac{p_t}{p_{it}} \right)^{\frac{1}{1-\lambda}}.$$

Discuss the form of this function when $\lambda \to 1$.

Suppose the i^{th} intermediate good is produced by a single firm using the following production function:

$$y_{it} = k_{it}^{\mu} l_{it}^{\nu}, \ \mu + \nu > 1.$$

The intermediate goods firm rents capital, k_{it}, and labor, l_{it}, in perfectly competitive markets at prices $p_t r_t$ and $p_t w_t$, respectively and seeks to maximize profits:

$$\pi_{it} = p_{it} y_{it} - p_t r_t k_{it} - p_t w_t l_{it}.$$

(b) Explain why the condition $\lambda(\mu + \nu) < 1$ guarantees that the i^{th} firm's first order conditions for k_{it} and l_{it} are sufficient for an optimum.

(c) Derive the i^{th} firm's first order conditions for k_{it} and l_{it}.

(d) Define the i^{th} intermediate firm's markup as its output price, p_{it}, divided by its marginal cost. Derive an expression for the markup in this model.

At date 0, the representative household maximizes

$$\sum_{t=0}^{\infty} \beta^t u(c_t, l_t), 0 < \beta < 1,$$

where u is bounded and strictly concave. The household must respect the sequence of budget constraints,

$$p_t \left(c_t + k_{t+1} - (1-\delta)k_t\right) \leq p_t r_t k_t + p_t w_t l_t + \int_0^1 \pi_{it} di, t \geq 0,$$

where $0 < \delta < 1$ is the depreciation rate on a unit of aggregate capital, and

$$k_t = \int_0^1 k_{it} di,$$

where k_{it} is the amount of capital rented to the i^{th} intermediate goods producer.

(e) Write out the first order conditions for the household.

(f) Define a sequence-of-markets competitive equilibrium for this economy. Show that we are free to normalize p_t to unity.

(g) In a *symmetric equilibrium*, intermediate firms all make the same choices. Show that a symmetric equilibrium has the property $p_{it} = p_t = 1$, for all $i \in (0, 1)$, and that, in this case, aggregate output is related to the aggregate inputs as follows:

$$y_t = k_t^{\mu} l_t^{\nu}.$$

(h) What is the share of output paid to labor and to capital in this economy. If these shares sum to less than one, then where does the rest go?

2. Consider the economy with many identical, perfectly competitive firms, each having the following production function:

$$y_t = A_t k_t^{\alpha} l_t^{1-\alpha}, \quad A_t = \bar{y}_t^{\gamma}, \gamma > 0, 0 < \alpha < 1,$$

where $\bar{y}_t$ denotes the economy-wide average level of output. Firms regard this as beyond their control. Suppose firms are perfect competitors in the output and factor markets and take prices as given.

(a) Write out firms' first order conditions for capital and labor. Suppose all firms make the same decisions. Develop an expression relating economy-wide average output, $\bar{y}_t$, to economy-wide average capital and labor.

Households maximize $\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$ subject to $c_t + k_{t+1} - (1 - \delta)k_t \leq r_t k_t + w_t l_t$. The resource constraint specifies that economy-wide consumption and gross investment cannot exceed economy-wide output.

(b) Define a sequence-of-markets equilibrium for this economy.

(c) What is the share of output paid to labor and to capital in this economy. If these shares sum to less than one, then where does the rest go?

(d) Discuss the similarities and differences between the model in this question and the one in the previous question.

3. Consider the sequence representation of the following two-sector planning problem:

$$\max_{\{c_t, i_t, k_{1,t+1}, k_{2,t+1}, l_{1,t}, l_{2,t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{array}{c} c_t \leq z_t F(k_{1,t}, l_{1,t}) \\ i_t \leq q_t z_t F(k_{2t}, l_{2t}) \end{array} .$$

Here, k_{it} and l_{it} are capital and labor allocated to sector i, $i = 1, 2$. Assume that factors can be freely moved between sectors, subject to:

$$k_t \equiv k_{1,t} + k_{2,t}, \;\; l_{1,t} + l_{2,t} = l,$$

where k_t is the aggregate stock of capital given at the beginning of time t and l is the (fixed) amount of labor effort supplied by households. Finally, we also require

$$k_{t+1}, c_t \geq 0, k_0 > 0$$

and the identity

$$k_{t+1} \leq (1 - \delta) k_t + i_t.$$

The sequences $\{z_t, q_t\}_{t=0}^{\infty}$ are exogenously given. It is assumed that u is continuously differentiable, strictly increasing, and strictly concave, that F is continuously differentiable, strictly increasing in both arguments, homogeneous of degree one, and strictly quasiconcave, and that $\delta, \beta \in (0, 1)$.

(a) Show that a necessary condition for optimization is $\frac{k_{1t}}{l_{1t}} = \frac{k_{2t}}{l_{2t}}$, and that this implies the constraint set above can be replaced by

$$\begin{array}{c} c_t + \frac{i_t}{q_t} \leq z_t F(k_t, l) \\ k_{t+1} = (1 - \delta) k_t + i_t \text{ and } k_0 > 0 \\ c_t, k_{t+1} > 0. \end{array}$$

(b) Assume that the two productivity change series follow:

$$z_t = \gamma_z^t \text{ and } q_t = \gamma_q^t,$$

where $\gamma_z \neq \gamma_q$ are each greater than one. Suppose F has a Cobb-Douglas form, and $u(c) = c^{1-\sigma}/(1-\sigma)$, $\sigma < 1$. Let a *steady-state growth path* be a situation in which c_t, k_t, and i_t are growing at a constant rate. Let γ_k and γ_c denote the steady-state growth rates of the capital stock and consumption, respectively. Develop equations that determine γ_k and γ_c as a function of the parameters of the problem.

(c) Scale the variables and show that the sequence planning problem can be expressed as a problem involving no growing variables.

(d) Express the non-growing, scaled, representation of the planning problem in functional equation form.

(e) Write the competitive equilibrium that corresponds to the scaled planning problem as a sequence of markets competitive equilibrium.

4. The typical household can engage in two types of activities: producing current output and studying at home. Although time spent studying at home sacrifices current production, it augments future output by increasing the household's future stock of human capital, k_{t+1}. The household has one unit of time available to split between home study and current production. Any given amount of human capital accumulation, k_{t+1}/k_t, leaves an amount of time, h_t, left over for producing current output, where $h_t = \phi(k_{t+1}/k_t)$. Here, ϕ is strictly decreasing, strictly concave, and continuously differentiable, with

$$\phi(1-\delta) = 1 \text{ for some } \delta \in (0,1),$$
$$\phi(1+\lambda) = 0 \text{ for some } \lambda > 0.$$

The variable, h_t, must satisfy $0 \leq h_t \leq 1$. This implies that the household cannot set k_{t+1}/k_t greater than $1+\lambda$ or less than $1-\delta$. The initial stock of human capital, $k_0 > 0$, is given.

A household's effective labor input into production is the product of its time and human capital: $h_t k_t$. Total output is related to effective labor input by

$$f(h_t k_t) = (h_t k_t)^\alpha, \;\; \alpha \in (0, 1).$$

The resource constraint for this economy is

$$c_t \leq f(h_t k_t),$$

and the initial level of human capital, k_0, is given. The utility value of a given sequence of consumption, c_t, is given by

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \text{ where } u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}, \;\; \gamma < 1.$$

We suppose that $\beta(1+\lambda)^{\alpha(1-\gamma)} < 1$.

(a) Let $v(k_0)$ denote the maximized value of the planning problem. Show that $v(k_0) = w k_0^{\alpha(1-\gamma)}$, where $-\infty < w < \infty$ (hint: write the sequence representation of the planning problem as one of maximizing utility with respect to ν_0, ν_1, ν_2, ... where $\nu_t = k_{t+1}/k_t$).

(b) Show that w solves a particular functional equation.

(c) Show that the optimal policy rule for this problem is $g(k) = \theta k$ for some $(1-\delta) < \theta < (1+\lambda)$.

(d) Suppose there are two separate economies, which differ only in how patient households are. In the more patient economy the discount rate is $\widetilde{\beta}$ and in the less patient economy, the discount rate is $\beta < \widetilde{\beta}$. Show that in the economy with more patient households, the growth rate of human capital is greater.